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Thermal History of the Earth with consideration of the Variable Thermal Conductivity of its Mantle

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(Received 1957 September 30)

Summary

Data about the abundance of the radioactive elements and the thermal conductivity of the Earth are in favour of the view that its initial temperature $T_0(r)$ must have been below the melting temperature. The results of the calculations of $T_0(r)$ are presented for the Earth considered as being formed through accumulation of particles of a gas-dust protoplanetary cloud.

The redistribution of radioactive elements is considered from uniform distribution to a layered one.

The matter of the mantle is considered as a dielectric possessing not only molecular conductivity but a radiative transfer of heat in the high temperature range.

The Green's function method and hydraulic analogy method are used for the calculations of the solution of the thermal conductivity equation.

It is shown that the increase of effective thermal conductivity due to a radiative transfer of energy at a great depth does not produce any essential losses of inner heat. The thermal conductivity of the Earth's upper layers is largely determined by its molecular component decreasing as the temperature rises and thereby forbidding considerable losses of inner heat. That leads to the conclusion that the Earth's thermal history may be regarded as secular heating of its interior with simultaneous slight cooling of its outer layers.

1. Introduction

Several methods for the estimation of the temperature inside the Earth exist at present. The possibility of determining the temperature from the degree of the increase of electrical conductivity with depth and from the velocities of seismic waves has been shown. However, only an investigation of the thermal history of

the Earth can give an answer to the question whether the Earth is undergoing the process of cooling or heating. A number of problems of the Earth's evolution are connected with this question.

Experimental data about the heat flow lost to space, the Earth's age, density distribution in it, content of radioactive elements in rocks, thermal conductivity and thermal capacity of rocks are used for the study of the thermal history of the Earth on the basis of the heat conduction equation. Moreover, definite suggestions regarding the initial distribution of temperature and the function of the distribution of the heat sources in the Earth are also introduced.

2. Initial conditions

The temperature of fusion, or of the solidification of the Earth's matter, is most frequently accepted as the initial temperature $T_0(r)$. On this assumption however any heat generation in the deep interior of the Earth, whence no heat is escaping, should be excluded. Otherwise the Earth mantle's temperature would now exceed the melting point and the mantle could never become solid independently. The heat transfer in the interior is assumed to take place mainly by conduction as it is in the paper of Jacobs & Allan (1954) or by some more extensive mechanism. The small thermal conductivity of the solid outer layer would forbid considerable losses of inner heat and heat would accumulate inside the Earth, raising the temperature higher than the melting point temperature.

There is no reason to believe that the mantle is quite free from radioactivity at present. The data about the heat flow on the oceanic and continental areas having granite basalt layers of different thickness (Revelle & Maxwell 1952) give a direct indication that such sources are present deep inside the Earth's mantle. The increased amount of data concerning the content of U, Th and K in various rocks and meteorites leads to the conclusion that radioactive elements are spread all over the Earth (Birch 1956, Verhoogen 1956). Even in the iron meteorites, with the composition of which the core of the Earth is usually identified, a quite definite concentration (Davis 1950, Paneth, Arroll & Jacob 1942, Starik & Shatz 1956) of them was discovered. In addition to that, the amount of iron in the Earth's core is often supposed now to be much less than was believed earlier (Bullen 1957). According to the hypothesis of Lodotchnikov (1939) and Ramsey (1949) the composition of the Earth's core differs very little from that of silicates. Hence we cannot completely exclude radioactive elements even from the Earth's core and cannot assume $T_0(r) = T(\text{melting})$.

Cosmogonical ideas represent one of the foundations for conclusions about the initial temperature and the distribution of radioactive elements. According to modern ideas the Earth and the planets were formed as a result of an accumulation of particles of a gas-dust protoplanetary cloud. The most complete explanation of the main features of the structure of the solar system from the point of view of this hypothesis is given in the papers by O. Y. Schmidt and his collaborators (Schmidt 1954, 1957, Levin 1956). These papers also lead to the conclusion that $T_0(r)$ was less than $T(\text{melting})$. The initial temperature of the Earth was caused by the following three processes: (a) impacts of particles against the surface of the growing Earth; (b) compression of the matter inside the Earth under the pressure of the growing outer layers; (c) radioactive heat, which could accumulate during the process of the Earth's formation if this process was not very short.

Effect (a) was estimated by Safronov (1954, 1958) for two limiting schemes of the Earth's growth: (1) at the expense of large particles, which are not mutually

colliding; and (2) at the expense of small inelastically colliding particles. If the variations in the particle velocity are not taken into aecount the maximum increase of temperature is about 200° K in the case of the first scheme and about 1200° K in the second scheme at a distance of about $0{\cdot}8R$ from the Earth's centre.

Taking into account the variation of the velocity of particles in the first case, which is most probable, the maximum temperature increase will only be 50–100° K. The period of the formation of the greatest part of the Earth (about 97 per cent of its mass) is about $0{\cdot}23.10^9$ years (Safronov 1958). A correction for radioactive heating for this period is about 700° K (if the Earth's age is about 5.10^9 years). The temperature rise due to the increased pressure in the process of the Earth's growth can be estimated from the formula (Lubimova 1955a)

$$T_0 = T_{00} \exp\left[\frac{1}{2a}\left(\frac{1}{\rho_0{}^2} - \frac{1}{\rho^2}\right)\right] \qquad (1)$$

where T_{00} is the temperature of the matter before its compression, a is the coefficient characterizing the dependence of Gruneisen's constant on the volume, and ρ_0 is the density of the matter before its compression. If all the above effects are taken into account the maximum temperature rise in the process of the Earth's formation is about 1200° K in its centre.

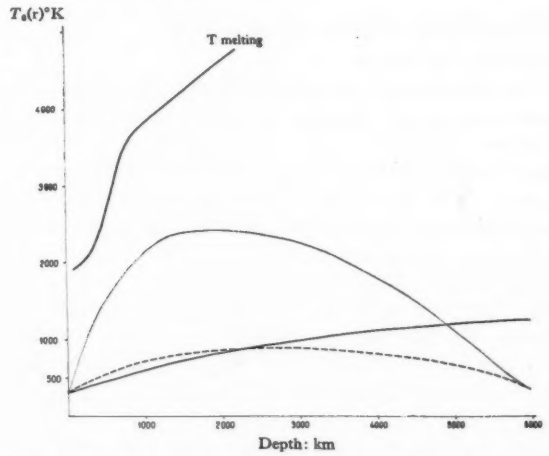

FIG. 1.—Distribution of the initial temperature $T_0(r)$ in the Earth formed as a result of accumulation of gas-dust particles of a protoplanetary cloud.

Curves of the distribution of the initial temperature $T_0(r)$ are given in Figure 1. The hatched and the dotted curves were calculated without taking the variations of the velocity of particles into account and neglecting the radiogenic heat in the period of the Earth's growth (the hatched curve corresponds to scheme (1) of the Earth's growth; the dotted—the scheme (2)). These corrections were taken into account in the preparation of the continuous curve. As it is seen from Figure 1, $T_0(r)$ is much less than the temperature of fusion (thick line), as given by Uffen (1952). Thus the Earth formed as a result of an accumulation of particles of the gas-dust protoplanetary cloud would not become fused.

The distribution of the radioactive elements in the Earth was usually assumed,

when the thermal regime of the Earth was calculated, not to vary in the process of the Earth's evolution (Slichter 1941, Urry 1949, Jacobs & Allan 1954).

However, modern geological ideas lead to the conclusion about the permanent formation of the Earth crust, the gradual growth of the continents by means of a partial remelting of an initially uniform meteoritic mantle and carrying of the easily fusible sialic constituents from the inside to the surface of the Earth, as a result of which the distribution of the radioactive elements would have been changed (Rubey 1951, Jeffreys 1956, Wilson 1954, Magnitsky 1955).

From this point of view the equality of the heat flow on the oceans and the continents may be understood. The total amount of sources must be the same in these parts of the Earth, but underneath the oceans the matter is less differentiated, the heat sources being distributed more uniformly than in the continental regions.

The suggestion about the uniform distribution of the radioactive sources of heat during the initial stage of the Earth's evolution follows naturally from the hypothesis of its origin as a result of an accumulation of particles from the protoplanetary cloud. In the process of the Earth's evolution there had been a gradual passage from a uniform distribution of sources to the modern layered distribution.

The period of the existence of the uniform distribution of sources can be assumed approximately equal to the difference between the values of the age of the Earth as a planet ($(4–5).10^9$ years) and of the age of the Earth's crust (3.10^9 years).

3. Heat conduction with constant conductivity

3.1 *Uniform distribution of sources*

A solution of the equation of heat conduction containing the function of sources $H(r, t)$ at constant κ, c, ρ was obtained by means of Green's function using the method of reflections (Lubimova 1952, 1956a, 1956b) for its construction. This method, developed in the mathematical papers of Tychonoff (1937) and Tychonoff & Sumarsky (1951), leads to a rapid convergence of series. After some transformations an expression for the temperature inside a spherical body is obtained in the form of a difference of two items having a simple meaning. The first depends upon the time and represents the temperature resulting from the heat generated in a given volume. The second, in the form of a rapidly converging series, characterizes the influence on the temperature of heat escape towards the surface*. For the thermal diffusivity of rocks $k = \kappa/c\rho = 0{\cdot}01$ cm^2/s (κ – thermal conductivity), and for time intervals less than 6.10^9 years it is sufficient to take the first term of the series, if the body radius is greater than 1600 km (for the Earth and the Moon in particular). From 4 to 16 terms are taken into account (Lubimova & Starkova 1954) for the radii of 200, 100, 50 km (the dimensions of the largest asteroids).

The temperature inside the Earth for a uniform distribution of the sources and of the zero initial† and the surface temperature is given by the formula

* The obviously applied Laplace transformation method (Lowan 1933) or of the division of variables (Allan 1955) led to extremely slowly converging series. Their summing up has become possible only since the electronic computers were introduced (Jacobs & Allan 1954). It may be shown that the method of automodel solutions leads to the same slowly converging series.

† The influence of the initial temperature $T_0(r)$ can be taken into account separately by means of an additional term representing the solution of the homogeneous equation of thermal conductivity for the cooling globe, deprived of internal sources of heat:

$$\frac{1}{r}\int_0^R T_0(r').G(r,t,r',0)dr$$

where $G(r, t, r', \tau)$ is the Green's function (Lubimova 1952, 1956).

$$T(r,t) = \frac{1}{c\rho}\int_0^t H(\tau)d\tau - \frac{R}{r\,.\,c\rho}\int_0^t H(\tau)\,.\,\Phi\left[\frac{(R-r)}{\sqrt{4k(t-\tau)}}\right]d\tau \qquad (2)$$

$$H(t) = \sum_a H_a(t_0)\exp(-\lambda_a t); \quad \Phi(x) = 1-\operatorname{erf} x \qquad (3)$$

where λ_e is the decay constant of the element a, and $H_a(t_0)$ is its rate of heat generation t_0 years ago. From formula (2) it follows that the region of heat escape towards the surface, and the inner region where all the emitted heat is spent on heating may be distinguished in the temperature field of the Earth. In the internal region $r < R-C(kt)^{\frac{1}{2}}$* the second term is negligible, the temperature being determined by the first term, increasing with the rise of t.

In the surface layer ($r < R-C(kt)^{\frac{1}{2}}$) the second term becomes comparable with the first one. Some portion of heat flows from this region towards the surface. The depth of the region of escape is proportional to $(kt)^{\frac{1}{2}}$. For a usually accepted value of $k = 0{\cdot}01$ cm^2/s it equals about 1000 km and may reach the centre after 10^{10} years only. It follows that the cooling of the Earth as a whole cannot begin earlier than after 10^{10} years. If $k = 0{\cdot}1$ the region of heat escape will exceed 2900 km already after 2.10^9 years of the Earth's formation (Lubimova 1953). Heat generation below the region of heat escape, as well as the changes of the physical properties of matter which are going on there, do not influence the heat flow observed at the Earth's surface.

The obtained solution permits us to calculate the thermal flow

$$q = \kappa\cdot\frac{\partial T}{\partial r}\bigg|_{r=R}$$

lost by the Earth. Results of a calculation of q for various values of the mean concentration of the radioactive sources and of the different values of age are given in Figure 3. The contents of U, Th and K in the Earth were taken equal to their mean abundances in meteorites according to the two sets of values A and B obtained earlier (Lubimova 1952). According to the value A one gramme of the Earth contains $0{\cdot}93.10^{-8}$ g of U, $0{\cdot}67.10^{-10}$ g of AcU, $2{\cdot}8.10^{-8}$ g Th and $1{\cdot}95.10^{-7}$ g of K^{40}. According to the value B: 1.10^{-7} g of U, $0{\cdot}72.10^{-9}$ g of AcU, 3.10^{-7} g of Th, $1{\cdot}95.10^{-7}$ g of K^{40}. A new value, C, is obtained from recent data (Starkova 1955): $5{\cdot}2.10^{-8}$ g U, $3{\cdot}7.10^{-10}$ g AcU, 21.10^{-8} g Th, $0{\cdot}9.10^{-7}$ g of K^{40}. This value C gives the curve of heat generation, which differs little from the curve of variant A for the last 2.10^9 years and practically coincides with it in the more remote past (Figure 2). All these values are within the intervals limiting on the one hand the minimum value introduced recently by Urey (1956) and on the other hand a maximum value, which may be deduced from the new data by Starik & Shatz (1956) (Figure 2).

We see from Figure 3 that the surface heat flow has decreased during the last 2000–3000 million years. However, conclusions about the cooling of the Earth as a whole, such as were supposed by Urry (1949), are not correct. The Earth's interior

* C is a coefficient depending upon $\sum_a H_a(t_0)$.

which is below the region of the heat escape is getting heated. Only the surface layers are cooling.

The concentration of radioactive elements, the age of the Earth and the lost heat flow are mutually connected. The calculated values of the flow q are within the range of the measured values $(1{\cdot}2 \pm 0{\cdot}6).10^{-6}$ cal/cm²s, if concentration A and the Earth age of $(4\text{–}5).10^9$ years are accepted (see Figure 3).

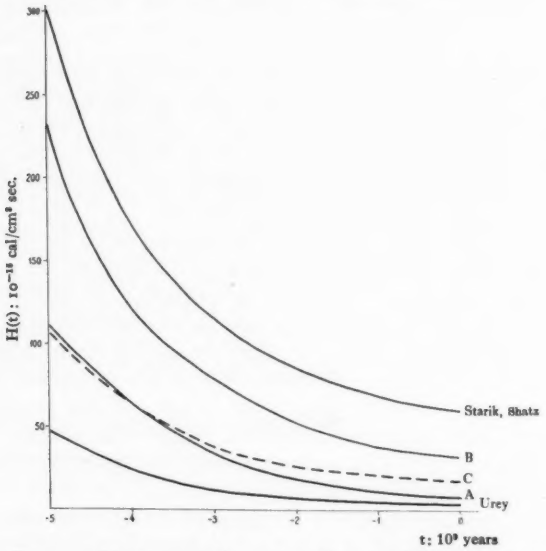

FIG. 2.—Heat generation, according to different values.

Formulae obtained for the temperature assuming a uniform distribution of heat sources are applicable to the description of the thermal regime of bodies having much smaller masses than the Earth, as for example the Moon and the asteroids. Differentiation of the matter of these bodies may be neglected. It may be considered in some approximation that these formulae describe also the thermal regime of the Earth below the oceans, where there is a very low differentiation of matter.

The features of the thermal field and of the heat flow of the Earth pointed out above are maintained, as it will be shown below, both for a non-uniform distribution of heat sources, as well as for a variable coefficient of thermal conductivity.

We suggested that the change from a uniform distribution of heat sources to their layer distribution took place 3.10^9 years ago. The temperature distribution established by the moment of the differentiation was accepted as the initial one. The total content of U, AcU, Th, K in the Earth corresponded to the variant A. The process of redistribution of sources is supposed to have taken place only in the mantle and the crust of the Earth; the region of the core was not involved. As a result of this the heat generation in the 20 km crust increased ten times (up to $1{\cdot}3.10^{-13}$ cal/cm²s), while in the mantle up to a depth of 2900 km it has correspondingly diminished 8 per cent.

3.2 *Layered distribution of sources*

The solution of the equation of heat conductivity for a layered distribution of heat sources was also obtained by means of the Green function with the aid of the

method of reflections (Lubimova 1956). Curves calculated from these formulae are given in Figures 4 and 5. The changes that took place before the differentiation were not taken into account in the curves of Figure 4. Curves of Figure 5 give the resulting temperature distribution*, in which the influence of the heat discharged in the stage of the Earth formation and the stage of uniform distribution of radioactive sources were taken into account. From these curves it is seen that the high concentration of the sources in the Earth's crust cause but an insignificant increase of its temperature, because the largest part of the heat generated in

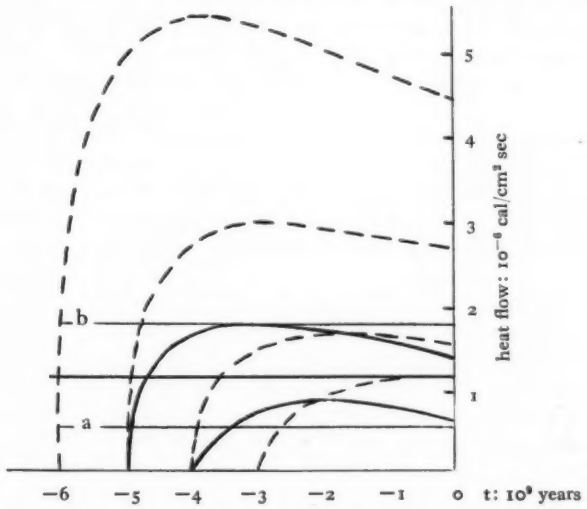

FIG. 3.—Surface flow of heat for the different Earth's ages and contents of U, Th, K. Continuous line is the variant *A*, hatched line is the variant *B*. Straight lines *a* and *b* are the limits of the interval of observed values *q*.

the crust was rapidly lost in space. The increase of the heat flow q was 20–30 per cent as compared with the flow at a uniform distribution of sources. At present the heat flow equals $1{\cdot}26.10^{-6}$ cal. per cm²/s which is close to the observed one. A comparatively small increase of q is explained by the fact that the dimension of the region of heat escape exceeds the thickness of the Earth's crust many times. We see that about 2.10^9 years ago the heat escape to the surface began to exceed the total amount of heat generated in the crust and carried out from the deeper layers. A lowering of the temperature of the upper layers has begun as a result of this.

The curves of Figure 5 confirm the conclusion that the thermal histories of the upper and internal layers of the Earth are different, which was arrived at from the simple instance of a uniform distribution of the heat sources. An accumulation of heat inside the Earth continues even at present. The same conclusion seems to be made recently by Jacobs & Allan (1954). The temperature of the base of the mantle has increased about 200° in the last billion years. The temperature of the mantle in this place is about 4000°C at present. Thus the thermal history of the Earth may be represented as a secular heating of its interior, rather than its secular cooling.

* For the calculation of the curves of Figure 5 the content of the radioactive elements was taken according to the variant A, the Earth's age equal to 4.10^9 years and the initial temperature $T_0(r)$ in accordance with the dotted line of Figure 1.

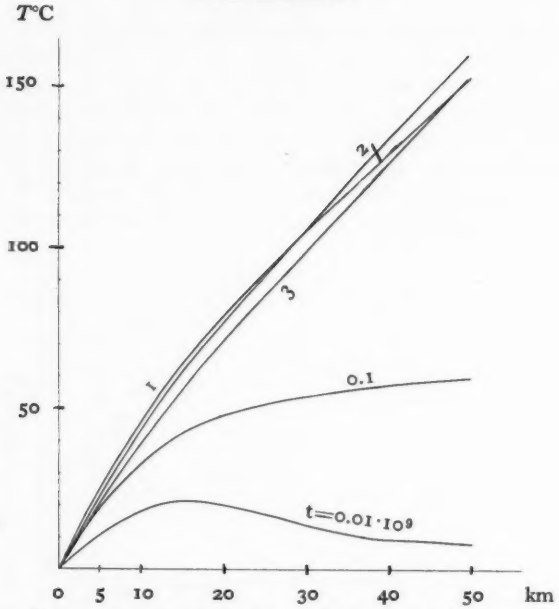

FIG. 4.—Temperature distribution near the surface of the layered Earth. ($T_0 = 0$).

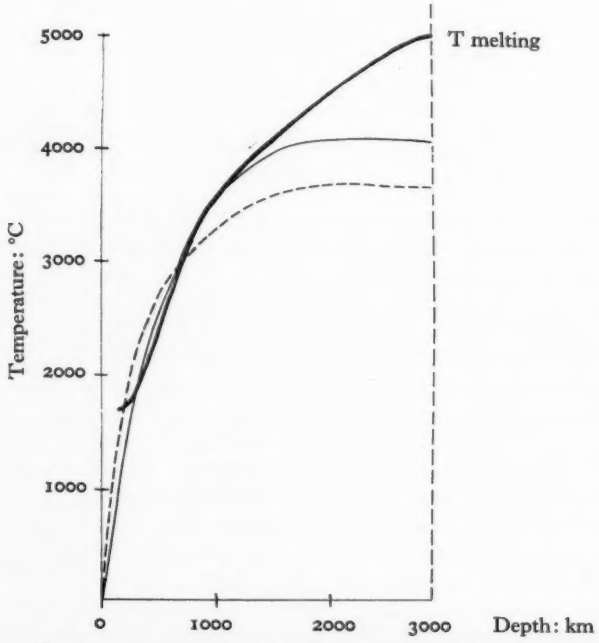

FIG. 5.—Temperature distribution in the Earth calculated at the assumption of constant coefficient of thermal diffusivity k m 0·01 cm²/s. Thin continuous line is the temperature at present, hatched line is the temperature 2·10; years ago, thick line is the melting temperature.

Another property of the temperature field of the Earth is the cooling of the upper layers, which are inside the zone of the heat escape.

4. Thermal conductivity of the Earth

The above conclusions depend essentially upon the thermal conductivity of the Earth, which determines the rate of the losses of heat by the Earth. Let us discuss this question.

According to its conducting properties the mantle of the Earth resembles dielectrics most closely. The thermal conductivity of dielectrics is determined by the vibrations of their crystal lattice. Therefore the estimates of the changes of thermal conductivity κ, obtained according to the data about electric conductivity (McNish 1937) proceeding from the Wiedemann-Franz law, are applicable only in the region of the metallic core of the Earth.

In order to obtain a better estimate of the thermal conductivity of the mantle one should proceed from theory of the lattice thermal conductivity of solids. Debye (1914) has shown that the main property of a crystal lattice leading to its finite thermal conductivity is the anharmonicity of vibrations of its atoms. The thermal motion of atoms must be represented as a totality of mutually interacting plane waves (phonons).

Peierls (1929, 1955) investigated this question from the point of view of quantum mechanics, taking the discrete character of the crystal lattice into account. Only the collisions of the phonons of a discrete lattice lead to a finite thermal conductivity, as a result of which the summary wave vector is changing; this change is equal to a period of the inverse lattice. These are the so-called "Umklapp" processes. At high temperatures, $T \gg \theta$ (θ—the Debye temperature) the "Umklapp" processes are as frequent as the ordinary collisions. At low temperatures $T \ll \theta$ the probability of such processes is small, the free path of the phonons being large; therefore the scattering of the phonons on the lattice defects, impurities, or mosaic blocks has a considerable influence on the thermal conductivity of dielectrics. In the present paper one high temperature case is of interest, when such effects are not appreciable. Debye and Peierls took the third-order terms into account in the expansion of the potential energy in a power series with respect to the displacement of atoms from their equilibrium positions. A similar dependence $\kappa \propto T^{-1}$ was obtained by them in the high temperature region $T > \theta$. The same dependence was obtained according to semi-empiric estimates by Dugdall & MacDonald (1955) and Kontorova (1956), who also restricted themselves to the approximation of the third order.

Pomeranchuk (1941, 1943) has shown that the temperature dependence of thermal conductivity obtained by Debye and Peierls takes place only for bodies having the special form of dispersion of velocity of thermal vibrations and special dependence of the velocity upon the direction. In the general case of an isotropic body anharmonicity alone can give no thermal resistance and the integral determining the thermal conductivity*

$$\kappa_1 \approx \int C_p u_p l_p \frac{d\mathbf{p}}{8\pi^3\hbar^3} \tag{4}$$

will be infinite unless fourth-order terms are included in the expansion of the potential energy, according to the powers of the displacement of the atoms from

* The symbols in formulae (4) are: l_p—the free path of a phonon, C_p—the specific heat of a phonon, u_p—its group velocity, $d\mathbf{p} = dp_x dp_y dp_z$; $\mathbf{p}/\hbar$—is its wave vector.

equilibrium positions, that is, collisions in which four phonons are participating are included.

The peculiarities of such collisions had been examined by Pomeranchuk (1941, 1943), who obtained a solution of the kinetic equation for this case.

The following expression was obtained for κ_1:

$$\kappa_1 \sim kNua\left(\frac{Mu^2}{T}\right)^{5/4} \tag{5}$$

where N is the number of elementary cells in 1 cm^3, k is the Boltzmann constant, M is the mass of a cell, a is the lattice constant, u is some mean sound velocity, which is determined from u_e the velocity of longitudinal elastic waves and u_t the velocity of transverse elastic waves by the formula

$$\frac{1}{u^3} = \frac{1}{3}\left(\frac{1}{u_e{}^3} + \frac{2}{u_t{}^3}\right).$$

The dependence of the thermal conductivity on pressure is determined according to formula (5) by means of values of u and a.

The matter of the Earth possesses isotropic velocities of sound; its temperature is higher than the Debye temperature. This gives ground for supposing that formula (5) may be applied to a description of the phonon component of thermal conductivity of the mantle.

Apart from ordinary conduction by means of lattice vibrations, the radiative transfer of heat, due to the scattering of electromagnetic waves, is possible in dielectrics at a very high temperature.

It has been established experimentally (McQuarrie 1954) that the thermal conductivity of ceramic materials of the MgO, Al_2O_3, BeO types decreases as $1/T^n$ where $n \doteqdot 1$, for temperatures $T < 1500$–1800°K. However at $T > 1500$–1800° it begins to rise rapidly. The same was established for glasses (Ginsburg 1930). At these temperatures along with the usual molecular thermal conductivity the radiative transfer is introduced.

Since the mantle of the Earth consists of substances similar to ceramics and glasses, the role of the radiative transfer of heat inside the Earth must be taken into account (Clark 1956, Preston 1956). Indeed the investigations of the olivine spectra (Keller, Spots & Briggs 1952, Runcorn 1956) show that the maximum radiation at temperatures prevailing in the Earth will take place in the interval of respective transparency.

As was shown by Van der Held (1952) the effect of radiative transfer in solids may be taken into account by introducing into the equation of thermal conductivity, instead of the usual coefficient of thermal conductivity, the coefficient consisting of the sum $\kappa = \kappa_1 + \kappa_2$ where, taking the radiative transfer of heat into account, κ_2 is

$$\kappa_2 = \frac{16}{3} \cdot \frac{n^2 \sigma T^3}{\epsilon}. \tag{6}$$

ϵ is the coefficient of the extinction of the material, n is the refractive index, σ is the Stefan-Boltzmann constant. Such a method of account follows from the transformation of the usual equations of radiative transfer in a grey body, provided that the solids possess a considerable absorption coefficient. There are grounds for

considering the matter of the Earth as a grey body, because of a high spreading of the olivine spectra.

As a whole the thermal conductivity of the Earth's mantle must be determined by the sum of phonon and radiative thermal conductivity:

$$\kappa = \kappa_1 + \kappa_2 = B\rho^{2/3}\frac{u^3 u^1}{T^{5/4}} + \frac{16}{3}\frac{n^2\sigma T^3}{\epsilon}. \tag{7}$$

The coefficient of proportionality B in (7) was determined by us from the condition that the molecular thermal conductivity κ_1 must be equal to the normal thermal conductivity of rocks $\kappa_1 = 0{\cdot}006$ cal/g.s.deg C on the Earth's surface at $u = 3{\cdot}6$ km/s, $\rho = 2{\cdot}7$ g/cm^3; $T = 277°$K. As it is seen from formula (7) the first term is decreasing with the rise of temperature, while the second term is increasing. This circumstance is very essential for the study of the thermal history of the Earth.

Clark found that with decreasing coefficient of extinction from $\epsilon = \infty$ to $\epsilon = 100 - 10^{-1}$ cm the temperature of the Earth's interior must fall 3 times. However the decrease of the molecular thermal conductivity κ_1 with temperature, which takes place in the upper layers and prevents considerable amounts of heat being carried out from the Earth's interior (owing to which no rapid cooling of the Earth can take place (Lubimova 1957)), was neglected by him.

The calculation by Clark was carried out on the assumption that temperature distribution is stationary in the Earth. In consequence of this he was obliged to take a 100 times less radioactive heat generation in the Earth, in order to satisfy the heat flow observed at the surface.

5. Heat conduction with variable conductivity

Let us consider the non-stationary equation of thermal conductivity

$$c\rho\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[(\kappa_1 + \kappa_2)r^2\frac{\partial T}{\partial r}\right] + \sum_a H_a \exp(-\lambda_a t). \tag{8}$$

It becomes nonlinear when the changes of the coefficient of thermal conductivity $\kappa = \kappa_1 + \kappa_2$ and of density $\rho(r)$ are taken into account. The method of hydraulic analogies of Lukyanov (1939) was used for the solution of this equation with the aid of the hydro-integrator. This apparatus permits the simulation of the process of thermal conductivity for variables κ, c, ρ in one, two and three-dimensional space. The change of the amount of heat in an elementary volume V_k cm^3, per time unit equal to $dQ/dt = c\rho V_k dT/dt$, is simulated by changing the quantity of water $d\bar{Q}/d\tau = \omega_k dA_k/d\tau$ where A_k is the level of water in a volume $\bar{V}_k$, ω_k the area of its cross-section in cm^2. A similar analogy is established for the heat flow $q = K(T_k - T_{k-1})$ from volume V_k to V_{k-1} and the water flow $\bar{q} = r_{k,k-1}(A_k - A_{k-1})$ between $\bar{V}_k$ and $\bar{V}_{k-1}$. K and $r_{k,k-1}$ are the coefficients of exchange, which are the values inverse to the resistances: the thermal value

$$R_{k,k-1} = \frac{1}{K} = \frac{d_{k,k-1}}{\kappa . F}$$

(κ—thermal conductivity, F—section of the area crossed by the heat flow q, $d_{k,k-1}$—the distance between the volume centres) and the hydraulic value:

$$1/\rho_{k,k-1} = r_{k,k-1}.$$

If the method of dimensions is used, the criterion of similitude is obtained, from which the scale of time m_τ and the law of simulation of heat generation follows from the formulae

$$m_\tau = \frac{c\rho R_{k,\,k-1}}{\omega_k \rho_{k,\,k-1}}; \quad \hat{\rho}_{k,\,k-1} = \frac{c\rho m_a m_H}{\omega_k \,.\, m_\tau}$$

where $\hat{\rho}_{k,k-1}$ is the hydraulic resistance determining the water flow, which simulates the heat generation; m_a and m_H are the scale of temperature and the scale of intensities of the internal sources respectively.

The precision of calculations by means of the hydro-integrator depends upon the degree of the division of the model into blocks. In our calculations the error did not exceed 7 per cent. It may be reduced to 1 per cent.

The calculation of the thermal history of the Earth, taking the variation of the thermal conductivity according to the depth into account, was carried out according to the scheme of its evolution given in the first part of the present paper. But the thickness of the Earth's crust was accepted as 40 km. The mean heat generation in the Earth's crust was calculated according to the data on the content of U, Th, K in the granites and basalts given by Birch (1954), which coincide approximately with those given by Bullard (1954). The ratio of granites and basalts in the crust was taken as 1 : 2. This leads to heat generation 3.10^{-13} cal/cm^{3}s at the present time. The initial temperature $T_0(r)$ was taken to correspond with the continuous curve of Figure 1 deduced on the ground of recent and more precise estimates. The mean abundance of radioactive elements was also taken from more precise data according to the estimate C (see above). The age of the Earth was taken equal to $4{\cdot}5.10^9$ years (Patterson, Brown, Tilton & Inghram 1953)*. The redistribution of radioactive sources of heat was supposed to take place about $3{\cdot}0.10^9$ years ago.

According to the data of a comparative analysis of the density distribution in the Earth and other planets of the same chemical composition (Koslovskaya 1953, Levin 1956) the differentiation of the matter in the Earth affects only the upper part of the mantle. We supposed in this connection for the last model that the carrying out of the sources into the Earth's crust took place in the upper part of the mantle to a depth of about 1400 km. The concentration of radioactive elements in the impoverished part of the mantle will correspond to a dunite one. Below 1400 km, including the region of the core, the source content was supposed similar to what it had been before the differentiation, corresponding namely to the mean of their content, C, in the iron and stone meteorites.

The phase transition on the boundary of the core must have been going on when the mass of the Earth reached 0·8 of its present mass (Ramsey 1949). We believe therefore that the core had metal conductivity already in the stage of uniform distribution of sources. The thermal conductivity of the Earth's core was calculated according to the law of Wiedemann-Franz.

$$\kappa = 3\left(\frac{k}{e}\right)^2 . T . \lambda$$

(k—Boltzmann constant, e—electron charge, λ—electric conductivity). κ was taken as 0·5 cal/deg C cm/s. As the change of λ in metals is inversely proportional to the temperature, thermal conductivity, κ, of the core does not depend upon T.

* If the period of the Earth's formation including the age of the Earth is equal to about $4{\cdot}7.10^9$ years.

The radiative transfer must have a very small effect in the Earth's core, as the absorption of radiative energy in metals is very great: ϵ reaches 10^5 cm^{-1}. In addition the absorption changes very little along the spectrum and does not depend consequently upon the temperature and pressure.

The most uncertain value is the coefficient of absorption ϵ, which is very sensitive to the increase of T and the radiation frequency. The preliminary value of it given by Clark is $\epsilon = 20$–40 cm^{-1} for olivine at room temperature at wavelengths of $0{\cdot}8\mu$. As may be judged from the data for semiconductors (Fan 1956) the coefficient of absorption increases with temperature. For T close to 600°K the coefficient of absorption in Si and Ge reaches $\epsilon = 100$ cm^{-1}.

We took $\epsilon = 200$ cm^{-1} previously (Lubimova 1957) and $\epsilon = 10$ cm^{-1} in the present paper. The refractive index was taken to be $n^2 = 3$ (about the value for olivine or MgO in the visible part of the spectrum).

6. Conclusion

Results of temperature calculations and the corresponding thermal conductivity distribution is given in Figures 6 and 7. We see that for all the moments of time thermal conductivity at first decreases with depth, then rapidly increases at great depth. The effective conductivity κ at the base of the Earth's mantle is four times larger than the ordinary conductivity of rocks if $\epsilon = 200$ cm^{-1} and 15 times if $\epsilon = 10$ cm^{-1}. Inside the Earth, whence no heat is escaping, the temperature rose rapidly with time, while in the first upper hundred kilometres, out of which the heat escapes to the surface, a much more gradual temperature increase was taking place. Radiative transfer is going on only very deep inside the mantle. The thermal conductivity of the upper layers is determined by the molecular component κ_1, owing to which total thermal conductivity κ has minimum equal to 0·003 cal/degC cm/s at a depth of 50–100 km. The value κ decreases at this depth by 1·5–2 times as compared with the normal thermal conductivity of the surface rocks. This decrease of thermal conductivity is followed by a number of peculiarities in temperature distribution. On the background of the general decrease of the temperature gradient with the approximation to the centre a small interval of depths up to 20 km (see Figure 6a) may be chosen, where there is an increase of the gradient.

The rapid increase of the temperature gradient over the depth 1000–15000 km (Figure 6) is caused by our schematic supposition about the sharp variation of the concentration of the radioactive sources.

The low thermal conductivity of the upper layers prevents considerable heat escape from the inside of the Earth. As a result the heating of the interior of the Earth is continuing up to the present moment. Thus the conclusion reached earlier is confirmed: the region of heat escape to the surface and the internal part which is heated may as before be separated. Although the thermal conductivity of the internal layers of the Earth is considerably higher than that of the surface layers (it equals 0·03 cal/deg C cm s at a depth of 1000 km if $\epsilon = 10$ cm^{-1}) no considerable heat escape takes place from the depths exceeding 500 km, as the temperature gradient there is much less than the gradient in the upper layers.

The change of the heat flow with time corresponding to the distribution of temperature in Figure 6 is given in Figure 8.

The straight line signifies the moment of differentiation. The maximum value of the heat flow reached $3{\cdot}4.10^{-6}$ cal/cm^{2}s and took place $2{\cdot}5.10^9$ years ago. The modern value equals $2{\cdot}3.10^{-6}$ cal/cm^{2}s, which exceeds twice the

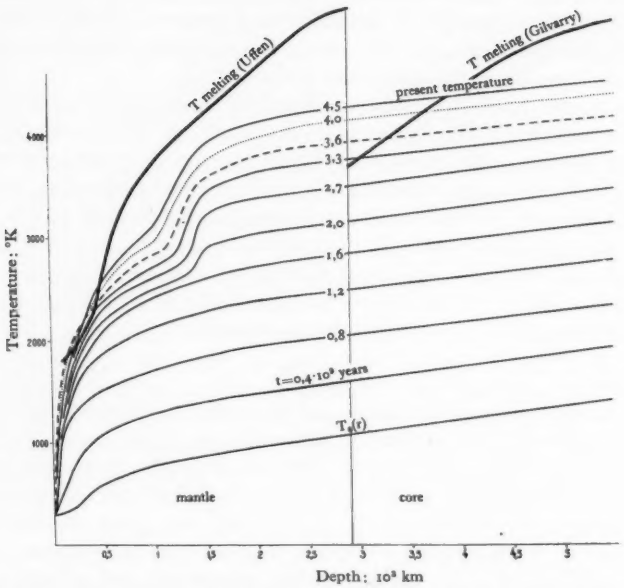

FIG. 6.—Temperature distribution in the Earth for different moments of time with account of the variations of thermal conductivity with depth.

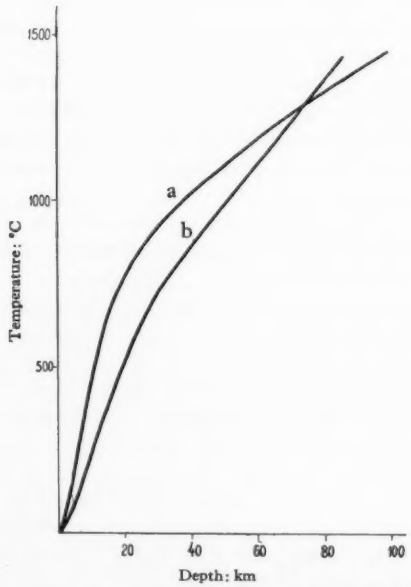

FIG. 6a.—Temperature in the upper layer of the Earth with account of variable thermal conductivity $\varkappa(r, t)$. *a*—from a study of thermal history of the Earth. *b*—extrapolation of the observed gradient with account of heat generation changes in granite, basalt and ultrabasalt layers.

observed mean value $1 \cdot 2 \cdot 10^{-6}$ cal/cm^2 s. This value was recently obtained by Clark and Niblett (1956) for the region of Alpine railroad tunnels. The heat flow obtained will be close to the mean observed for half the heat generation in the crust such as that given in Figure 5. This leads to a suggestion that the uppermost layer of the Earth cannot consist of a continuous granite massive only; the content of granites must be less. It agrees with Birch's (1955) idea about mosaic structure of the crust.

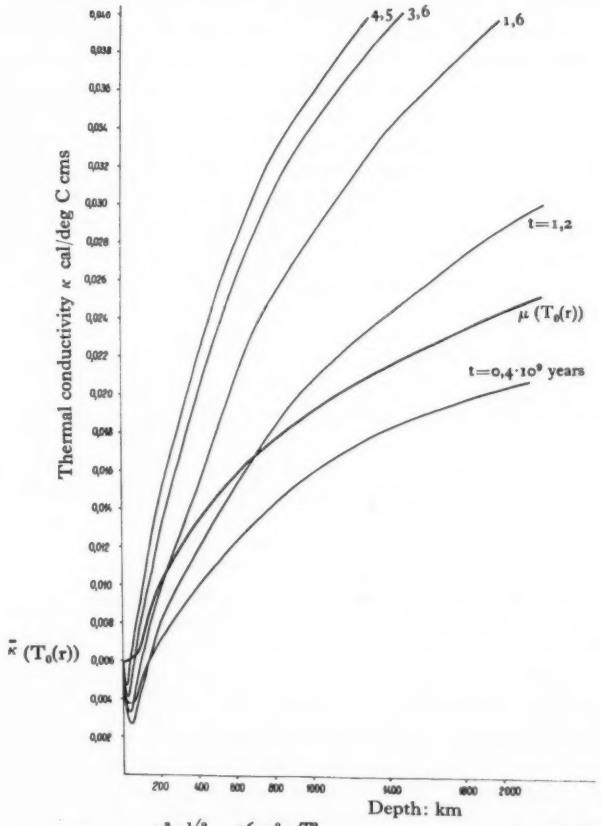

FIG. 7.—Distribution of thermal conductivity in the Earth's mantle.

The selection of the correct heat generation in the uppermost layers of the Earth is connected with the problem of granite distribution.

The depth of the region of heat escape is about 500 km. The temperature inside this has decreased about 100°C in the last 10^9 years.

We shall compare the temperature at various depths with Uffen's curve of fusion (see Figures 6 and 5). The temperature of the Earth increases rapidly in the upper layers after which its rise is lessened.

Contrariwise, the fusion temperature of the Earth rises slowly in the upper layers, in which the pressure is not large and rapidly in the inner layers. This leads to a tendency to fusion in the upper part of the Earth's mantle. This tendency

can also be noticed in the curves of Figure 5. The molten stratum formed in limits from 100–200 up to 500–700 km. Some suggestions may be made as to consequences of the appearance of the stratum. This melting might have been the cause of the extensive magmatic activity in the past which, according to certain ideas, led to the formation of the modern Earth's crust.

Differentiation of matter in the stratum under consideration must have resulted in reduction of heat sources in the lower part of the stratum and in their increase

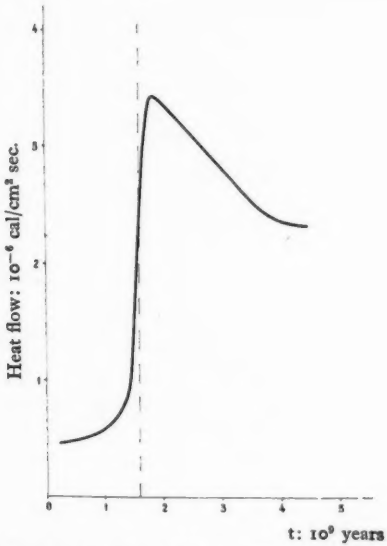

FIG. 8.—Heat flow variation with time after the redistribution of heat sources at the time $t_0 = 1{\cdot}6.10^9$ years.

at the top of it; it may be connected with the nature of the C-layer and the layer of low u_e and u_t velocities in asthenosphere. Movement of sources towards the surface must have led to decrease of temperature gradient at the base of the stratum and its increase at the top of this, which would explain the observed increase of u_e and u_t velocities and density ρ in the C-layer as well as decrease u_ρ, u_t in asthenosphere at depth about 150 km. Some accumulation of iron at the lower part of the stratum due to the differentiation must have raised the density too. Moreover, it may explain the observed rise of electrical conductivity in the C-layer.

It is feasible that the modern magmatic reservoirs are individually preserved oases of melting at the upper boundary of this layer. As to the lower boundary, as it is seen from Figures 5 and 6, the zone of the reservoirs is displaced with time towards the inside of the Earth. Its limiting depth is 500–700 km for the present. This is also the limiting depth for deep focus earthquakes.

The Earth as a whole could not melt in the process of its evolution. Complete melting is only possible for bodies of much lesser mass than the mass of the Earth, such as the Moon for example (Levin & Lubimova 1955). However, in the case of smaller bodies like the asteroids the melting could take place only in the centre (Lubimova & Starkova 1954). If the melting temperature for the metallic state of silicates is supposed to be close to that of iron, as estimated by Simon (1953) with

Gilvarry's correction (1957) for the pressure in the Earth's centre, then, as is seen from Figure 6, the outer parts of the core must be in a molten state, while the internal core must be solid. This agrees with the hypothesis of Jacobs (1954) about the rigidity of the inner core.

The general conclusion of this paper is as follows. The wide distribution of radioactive elements, which is difficult to deny, especially with reference to potassium, and the small thermal conductivity of the Earth's upper layers which can only decrease with the rise of temperature, must result in heat accumulation in the Earth's interior. But the rise of temperature proceeded so slowly that no catastrophe could occur and it only gave rise to slight expansion of the globe. The increment of the Earth's radius is about 7 cm for 1,000 years.

It is significant that the outer layers were not heated but rather cooled down during the last $(1–2).10^9$ years. The resulting compression of the surface layers covering the expanding bedding may be one of the causes of the formation of elastic strains in the upper part of the mantle which in turn result in faults and earthquakes. The possible appearance of a molten stratum in the upper part of the Earth's mantle below the crust gives one possibility of explaining the non-uniformity of this part and the origin of magmatic reservoirs.

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Note added in proof.—An article by Sydney P. Clark on "Radiative transfer in the Earth's mantle" appeared in *Trans. Amer. Geophys. Union*, **38**, No. 6, 1957, while I was reading the proofs of the present paper. In that article Dr Clark comes to the conclusion that "the presence of an outermost 600 km shell having a high conductivity (due to radiative transfer) may increase the heat loss from the Earth". He does not take into account, however, that thermal conductivity of the upper first hundred kilometers is determined by its molecular component, which decreases as the temperature rises and thereby prevents considerable heat losses in spite of high conductivity of the Earth's interior.

1958 *February.*

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Seismic Observations from the 1956 Atomic Explosions in Australia

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Summary

A discussion is given of seismic observations from the four 1956 British atomic explosions, at Maralinga, in South Australia. Both P and S phases were observed at field stations set up along the Trans-Australian railway for $0{\cdot}4° < \Delta < 11°$, and on a Benioff seismograph located near Adelaide at 8°.

The travel-times indicate a P_n velocity of $8{\cdot}21 \pm 0{\cdot}005$ km/s and an S_n velocity of $4{\cdot}75 \pm 0{\cdot}01$ km/s. There is also evidence for P and S phases travelling near the surface with velocities of $6{\cdot}03 \pm 0{\cdot}09$ km/s and $3{\cdot}55 \pm 0{\cdot}04$ km/s respectively. No onsets corresponding to paths through intermediate layers were observed.

The observations can be explained by a single crustal layer having constant seismic velocities. This hypothesis gives a crustal thickness of 32 km from P and 39 km from S.

The paper contains a more detailed treatment of the Maralinga data previously discussed by Doyle. These data have provided the first published studies of Australian crustal structure based on refraction methods.

1. Introduction

Early in 1956 it was announced that a number of British atomic bombs would be exploded at the testing range, near Maralinga, South Australia. Useful seismological information had previously been obtained from atomic tests in the U.S.A., the Pacific and Australia so, at the suggestion of Sir Edward Bullard and with the co-operation of the British Atomic Weapons Research Establishment, a programme was developed in Australia to obtain further seismic recordings from these explosions.

The programme was in three parts. The first part, undertaken by the Department of Geophysics, Australian National University, consisted of establishing field stations along the Trans-Australian Railway, west of Maralinga. Sir Edward Bullard originally suggested investigating the 20° discontinuity, but as the chances of recording to this distance were small, Professor J. C. Jaeger suggested recording along the railway towards Perth ($\Delta = 15°$). Those involved in establishing and

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operating these stations were H. A. Doyle, R. Green, H. Angus and E. Irving (all of the Australian National University), and B. A. Bolt (of the University of Sydney). As the second part, the Bureau of Mineral Resources undertook more detailed seismic recording within about 1° of the explosions. The third part involved the five permanent Australian seismological observatories, Brisbane, Riverview, Melbourne, Adelaide and Perth. Their positions relative to Maralinga may be seen on Figure 1. Arrangements were made to inform the Observatory directors of the explosion times in advance, through Professor K. E. Bullen. A Benioff seismograph was specially installed at Adelaide, by Dr Sutton, to replace the Milne-Shaw instrument and Dr J. Webb of the University of Queensland set up a field station at Quilpie (Qld.).

Atomic explosions occurred on September 27, October 4, October 11, and October 21, 1956; these will be called explosions A, B, C and D. Seismic waves were recorded at the seven field stations shown in Figure 1, and by the Benioff instrument at Adelaide, but none were recorded at Brisbane, Quilpie, Riverview, Melbourne or Perth.

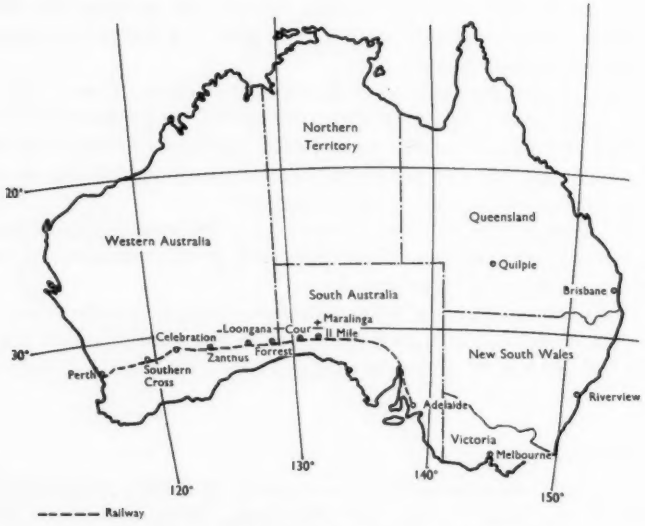


FIG. 1.—Maralinga and the recording stations.

In this paper, the observations from the field stations and Adelaide are discussed; those of the Bureau of Mineral Resources are not included. A preliminary study of the field observations has been given by Doyle (1957). Prior to these works no estimates of seismic velocities in Australia, comparable with studies in Europe, Japan, North America, South Africa and elsewhere, have been published. Enough detail is therefore included in the present paper to allow the uncertainties to be independently assessed.

2. The field stations

Four high-speed recorders and a Willmore seismograph were used. The high-speed instruments were built at the Bernard Price Institute, Johannesburg,

for seismic work in South Africa, and were kindly made available by the Institute for this experiment. They were used with a vertical and a horizontal moving-coil geophone (the Forrest instrument had an extra vertical geophone), of natural periods about 0·6 s. These were connected through battery-powered amplifiers to a galvanometer recorder, with paper speed 10–20 mm/s. A third galvanometer was connected to a chronometer which marked half-seconds and minutes.

Accurate clock corrections were obtained by recording time signal broadcasts on one of the galvanometers before and after the expected explosions. Time signals from WWVH (Honolulu) on 15 or 10 Mc/s, and VHP (Belconnen, Mt. Stromlo) on 6·4 and 8·4 Mc/s and 44 kc/s were found the most useful. The cessation and return of the tone on WWVH could often be read on the record within 0·05 s even when the pips were obscured by static. Dr A. R. Hogg, the Acting Commonwealth Astronomer, informed us that, because of travel delay and errors in adopted longitudes, WWVH signals would be about 0·07 s slow compared with Belconnen. Since the times of the explosion were known to only 0·1 s this correction was ignored.

The Willmore instrument was used with a vertical and a horizontal geophone, of natural periods 1 and 1½ s respectively. Minute marks from a chronometer were identified by means of hand pulses coinciding with the radio signals. As a check, chronometers at all stations were rated by ear each day. This check showed that the rate of the Willmore chronometer was not constant during the experiment. Corrections from a graph allowed times to be found within 0·2 s, which is less precise by a factor of four than the times from the fast recorders.

The field stations were established in buildings, kindly made available by the Commonwealth Railways, in the settlements along the Trans-Australian line The railway telephone system provided convenient communication between stations. The Zanthus, Celebration and Southern Cross sites were situated directly on the Yilgarn Pre-Cambrian Shield, while the remainder were on the Nullarbor Plain. This plain consists of horizontally bedded Miocene limestones which, from the evidence of boreholes, are probably no more than a thousand feet thick.

After explosion A, during which five field stations were operating, the number of observers was reduced. For subsequent explosions, the remaining observers either recorded at stations at which satisfactory results had not been obtained or established new stations.

3. The Mt. Bonython (Adelaide) Station

A three component, Benioff short-period moving-coil seismograph, with film recorder, was used. Initially the seismograph was located on piers in the basement of the observatory in the grounds of the University of Adelaide. However because of local noise it was moved for explosion D to the concrete floor of a shed at Mount Bonython, 11 miles from the city.

The stratigraphical sequence for Mt. Bonython indicates the presence of up to 6 000 feet of consolidated sedimentary rocks below the site; the height above sea. level is 2 200 feet.

Time pips at half-minute intervals, obtained from the South Australian Standard Time Service, were transmitted to the site by land-line. These pips differed from Belconnen time by less than 50 ms.

4. Observations

Details of the records obtained are shown in Table 1. The paper speeds are given only as orders of magnitude.

Table 1

Particulars of Records

Station	Instrument	Components	Explosion	Paper Speed mm/s.
11 Mile Camp	B.P.I.(1)	Z, H(2)	C	10
Cook	,,	Z	A	20
Forrest	,,	Z	B	10
Loongana	Willmore	Z, H(2)	A	1
Zanthus	B.P.I.	Z, H(2)	A	20
Adelaide	Benioff	Z, 2H	D	0·3
Celebration	B.P.I.	Z, H(2)	C, D	10
Southern Cross	,,	Z, H(2)	D	20

(1) High-speed recorder built by Bernard Price Institute.
(2) The horizontal geophones were orientated roughly towards Maralinga.

The co-ordinates and origin-times of the four explosions were kindly supplied by the Atomic Weapons Research Establishment.

The nearest recording was obtained at 11 Mile Camp using Explosion C. The record is very clear (Figure 3) but in this test the atomic device was dropped from an aircraft, causing higher uncertainty than usual in the location and time of the explosion. The supplied angular distance and explosion time have accuracies of $\pm 0°{\cdot}01$ and ± 1 s, respectively. The combined error for one observation is, however, unlikely to exceed 1·5 s which is the usual value of the standard error of reliable *P* observations from earthquake work (Jeffreys 1947), and the observations were useful for comparison with the calculated travel-times.

The co-ordinates and times of the remaining explosions have accuracies within $\pm 0{\cdot}1$ minutes of arc and $\pm 0{\cdot}1$ s respectively. The positions of the field stations were obtained from railway survey data and Department of Lands maps, and their accuracy is $\pm 0°{\cdot}005$. Angular distances Δ were calculated from the geocentric co-ordinates with six-figure tables using $2(1-\cos\Delta)$. Bullen (1934) has shown that the resultant maximum error is $0°{\cdot}034$ for $1° \leqslant \Delta \leqslant 20°$. The full ellipticity and height corrections are less than 0·05 s for all observations and were ignored.

Each author independently examined the records, reading only the prominent phases, and calculating the time corrections. On comparison, the three sets of travel-times were in striking agreement, only a few readings differing by more than 0·2 s. Those readings supported by only one of the authors were omitted; the remainder are shown in Table 2.

Table 2

Arrival Times of Phases

Station	Δ deg.	Travel-time s	Phase	Weight	Residual s
11 Mile Camp	0·47	8·2	P_1	(3)*	−0·45
		14·9	S_1	(3)	+0·20
		151·3	Sound wave	3	
Cook	1·32	24·0	P_1	3	−0·26
		25·7	P_n	3	+0·59 [+0·03]
		41·9	S_1	3	+0·68
		44·9	S_n	2	−0·24
		60·0		2	
Forrest	3·19	50·4	P_n	3	−0·03 [−0·26]
		58·0	P_1	3	−0·76
		62·1		3	
		89·5	S_n	3	+0·54
		97·6	S_1	3	−2·24
Loongana	4·13	63·2	P_n	3	+0·04 [−0·12]
		72·0		2	
		82·0	P_1?	1	+5·77
		111·2	S_n	3	+0·04
		131·5	S_1	2	+1·97
Zanthus	7·08	103·1	P_n	3	−0·00 [+0·41]
		110·8		2	
		132·1	P_1	2	+1·53
		179·9	S_n	3	−0·29
		222·4	S_1	2	+0·52
Mt. Bonython	7·85	116·2	P_n	3	+2·67 [+3·14]
		203·5	S_n	3	+5·22
		213·0		2	
		248·5	S_1	3	+2·29
Celebration	8·72				
Explosion D		125·3	P_n	3	−0·01 [+0·73]
		219·4	S_n	3	+0·45
		272·0	S_1	3	−1·20
Explosion C	±	126·5	P_n?	(1)	+1·19 [+1·93]
		217·8	S_n	2	−0·75
Southern Cross	10·69	150·2	P_n	3	−1·88 [−0·76]
		264·4	S_n	3	−0·43

*() denotes observations, which although weighted, have not been included in the formal calculations.

A simple system of weighting has been adopted based on the degree of agreement in the selection of phase. Weight 3 is given to all onsets agreed on within 0·2 s by the three authors; weight 2 is allotted when only two authors agree within 0·2 s; weight 1 indicates a spread of up to 1 s in the independently adopted arrival time of the phase.

The records of the fast recorders were the simplest to interpret, showing little complexity after the first arrivals of the main phases. This is probably due to the shorter period of the instruments. The vertical components had the larger amplitudes and the sharper onsets but the horizontal component provided a useful check. The Willmore records had many secondary onsets with often no agreement between vertical and horizontal components. This has also been commented on by Hodgson (1953). The Benioff instrument gave clear records showing small amplitudes, the largest being on the vertical component. (Although the first P onset was small the reading is not in doubt.)

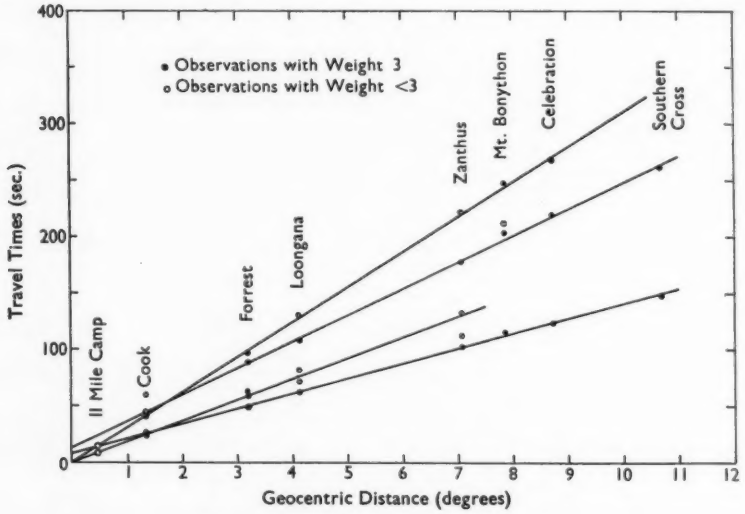
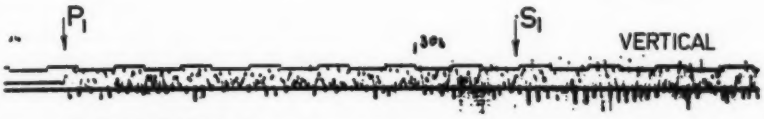


FIG. 2.—Travel-times for all stations.

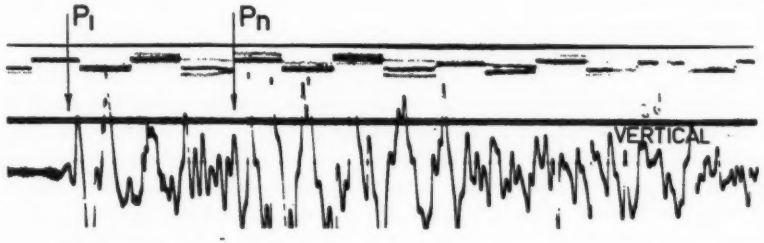
The travel-times are shown plotted against distance in Figure 2. This strongly indicates that there are four phases with characteristic velocities over the profile. These phases have been called P_n, P_1, S_n, S_1 in the order which corresponds to the order of arrival beyond Cook. Examples of the phases are reproduced in Figure 3.

In a similar way to the observations from the Heligoland explosion, reported by Willmore (1949), the first arrivals are uniformly well recorded over the whole range. In this respect the present observations are also in agreement with Tatel and Tuve's findings from blasts in Tennessee and in disagreement with Gutenberg's suggestion, based on Californian earthquake studies, of a shadow zone for P_n for $7° < \Delta < 13°$ (cf. Byerly 1956).

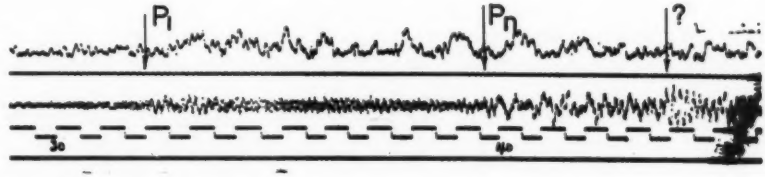
The Heligoland S observations are, however, dissimilar. Willmore states that transverse waves were poorly recorded by the short period vertical instruments; in the Maralinga experiment the S phases were generally comparable to the P in amplitude but had slightly longer periods (0·1–0·2 s). Gutenberg (1946, 1952) has drawn attention to S phases recorded from American atomic tests. From the 1945 New Mexico test they were recorded with indistinct beginnings and small amplitudes at observatories having $3°{\cdot}9 \leqslant \Delta \leqslant 9°{\cdot}7$; from the Nevada tests they were recorded up to $\Delta = 3°{\cdot}6$. In the latter case Gutenberg remarks that it is possible that some of the S phases are surface waves.



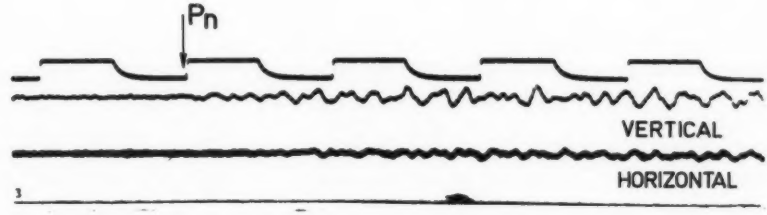
I 11 MILE CAMP $\Delta = 0°{\cdot}47$



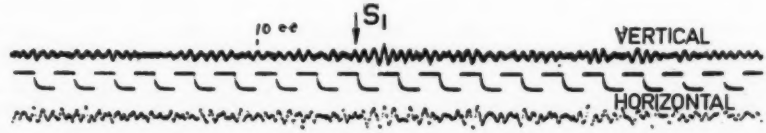
II COOK $\Delta = 1°{\cdot}32$



III FORREST $\Delta = 3°{\cdot}19$



IV ZANTHUS $\Delta = 7°{\cdot}08$



V CELEBRATION $\Delta = 8°{\cdot}72$

FIG. 3.—Typical Records.

5. Travel-times

Least squares solutions for the travel-times of P_n, P_1, S_n, S_1 were made using linear equations and allotting weights as indicated in Table 2. Equations for the Mt. Bonython and 11 Mile Camp observations were not included since Mt. Bonython is in roughly the opposite azimuth to the field stations and because of the relatively large uncertainty in the angular distance between explosion C and 11 Mile Camp.

For P_n, using only the first arrivals at Forrest, Loongana, Zanthus and Celebration, the result is:

$$t = (7{\cdot}24 \pm 0{\cdot}05)\,\text{s} + (13{\cdot}54 \pm 0{\cdot}01)\,\text{s/deg}.$$

The P_n surface velocity is $(8{\cdot}21 \pm 0{\cdot}005)$ km/s and the standard error of one observation $\sigma = 0{\cdot}04$ s. This gives residuals of $+0{\cdot}59$ and $-1{\cdot}88$ s for Cook and Southern Cross respectively. The Cook residual is quite satisfactory considering that the phase is a second arrival, but the Southern Cross residual is rather large and suggests the inclusion of a cubic term in the travel-time equation. An alternative solution, obtained by including equations for Cook and Southern Cross gives:

$$t = (8{\cdot}10 \pm 0{\cdot}49)\,\text{s} + (13{\cdot}36 \pm 0{\cdot}07)\,\text{s/deg}.$$

The corresponding velocity is $(8{\cdot}32 \pm 0{\cdot}04)$ km/s σ and increases to 0·58 s which is somewhat high for clear P_n arrivals measured independently by the three authors to within 0·2 s. The residuals from this alternative solution are included in Table 2 in brackets.

For distances up to 14° the J.B. times include a cubic term, approximately $-0{\cdot}0025(\Delta - 1)^3$, which amounts to $-1{\cdot}14$ s at 8°·7 (Celebration) and $-2{\cdot}3$ s at 10°·7 (Southern Cross). These values may be compared with the residuals from the first solution, $-0{\cdot}01$ and $-1{\cdot}88$ s respectively. Jeffreys (1952) has found that the times in Japanese earthquakes agree reasonably well with the J.B. cubic term, while times in Europe, and less certainly California, are consistent with linear formulae.

Observations marked P_1 in Table 2, from Cook, Forrest, Loongana and Zanthus give $t = 0{\cdot}72\,\text{s} + 18{\cdot}43\Delta$ which makes the Loongana residual $+5{\cdot}1$ s. The complexity of the Willmore record after the first arrival makes the identification somewhat uncertain so the Loongana equation was omitted, giving $t = 0{\cdot}90\,\text{s} + 18{\cdot}71\Delta$ with $\sigma = 0{\cdot}88$ s. The constant term therefore is not significant and we take it to be zero. This is equivalent to assuming a single-layered crust having constant P velocity. The solution is

$$t = (18{\cdot}45 \pm 0{\cdot}28)\,\text{s/deg}.$$

The P_1 residuals in Table 2 are obtained using this formula; $\sigma = 1{\cdot}49$ s. The surface velocity is $(6{\cdot}03 \pm 0{\cdot}09)$ km/s. Support for this solution comes from the first arrival at 11 Mile Camp, which has a residual of $-0{\cdot}45$ s. We note that on this solution P_n would not appear before 75 km, so the second arrival at 11 Mile Camp is not P_n.

There is no difficulty in combining the S_n observations from Cook, Forrest, Loongana, Zanthus, Celebration and Southern Cross. The solution is

$$t = (14{\cdot}33 \pm 0{\cdot}46)\,\text{s} + (23{\cdot}43 \pm 0{\cdot}06)\,\text{s/deg},$$

corresponding to a velocity of $(4{\cdot}75 \pm 0{\cdot}01)$ km/s over the surface. The standard

error for one S_n time is 0·48 s. As with P_n, the residual at Southern Cross is negative (−0·4 s). The J.B. formula for S_n at the distances involved here (Jeffreys, 1939) contains $-0{\cdot}00314(\Delta-1)^3$ which contributes −3·1 s at 11° and is therefore inconsistent with the Maralinga data.

The five S_1 observations at Cook, Forrest, Loongana, Zanthus and Celebration provide a check on our hypothesis of a uniform single-layered crust. We obtain

$$t = (1{\cdot}76 \pm 1{\cdot}94)\,\text{s} + (31{\cdot}00 \pm 0{\cdot}36)\,\text{s/deg}.$$

with $\sigma = 1{\cdot}97$ s. This standard deviation is of the order of the standard error of a good S_n observation from earthquakes (Jeffreys 1939), but unexpectedly high for a study with known distances and origin times. Until further information becomes available it seems best to suppose the observations are the same phase. The constant term may then be dropped to give

$$t = (31{\cdot}35 \pm 0{\cdot}40)\,\text{s/deg},$$

corresponding to a surface velocity (3·55 ± 0·04) km/s. The residuals against this solution are given in Table 2; $\sigma = 1{\cdot}80$ s. The reduction in σ is due to reduction in the large unexplained residual at Forrest. The residual of +0·2 s for the second arrival at 11 Mile Camp agrees with the solution and we conclude that the S_1 data is consistent with a single-layered crust.

The Mt. Bonython observations now give the following residuals, P_n +2·7 s, S_n +5·2 s, and S_1 +2·3 s. Delays of this magnitude are rather surprising. Explanation of the P_n and S_n residuals in terms of large down-faulted blocks in the Adelaide "shatter zone" leaves the S_1 residual unexplained. It seems best to wait for further data from Mt. Bonython for a final answer.

6. Discussion

The velocities and adopted crustal structure give the following crustal thicknesses:

From P waves	32·2 km.
From S waves	38·8 km.

The various standard errors imply uncertainties of order 3 km for both estimates. Hodgson (1953) and others have similarly obtained a higher value for the crustal thickness using S.

The phases P_1 and S_1 are of particular interest. Identification of P_1 is somewhat in doubt at stations beyond Forrest, while S_1 is clear at $\Delta = 8°{\cdot}72$ (Celebration) and its presence on the Southern Cross record may have been obscured by the high noise level at the time. In their study of the South Australian earthquake, 1939 March 26, situated about 10° east of Maralinga, Bullen & Bolt (1956) identified two prominent phases on the Adelaide record. These arrived 12 s and 48 s after P_n and were tentatively called P^* and S^*. The P_1-P_n, S_1-S_n intervals from the Maralinga travel-time curves at the same distance, $\Delta = 3°{\cdot}0$, are 8 s and 47 s respectively. The agreement between S^* and S_1 is an interesting feature since the S^* was thought to travel in an intermediate layer.

In this paper the P_1 and S_1 phases are taken to be waves travelling close to the surface. There is some indication that they correspond to the longitudinal and transverse waves Π_g and L_g. The L_g phase has been found to be efficiently propagated in all continental crusts with a mean velocity close to that found in North

America (Press & Ewing 1952), namely $(3{\cdot}51 \pm 0{\cdot}07)$ km/s. For Australia, Bolt (1957) has found a mean L_g velocity of $(3{\cdot}50 \pm 0{\cdot}07)$ km/s. The Π_g phase has been observed by Press and Gutenberg (1956) on seismograms of a North American earthquake; it is characterised by small amplitudes and erratic occurrence. They obtain a mean velocity of $(6{\cdot}09 \pm 0{\cdot}06)$ km/s from 11 observations with $5° < \Delta < 38°$.

There are a few onsets listed in Table 2 which are not explained. The second P phase at Loongana may be PP, while the second S phase at Mt. Bonython may be SS.

A large number of studies of travel-times from blasts in other countries are available for comparison with the Maralinga results. Of special interest is their agreement, set out below, with the results of Hodgson (1953) in the Canadian Shield, and with the results of Willmore, Hales & Gane (1952) in the Western Transvaal.

	Canadian Shield	*Western Transvaal*	*Western Australia*
P_n (km/s)	$8{\cdot}18 \pm 0{\cdot}01$	$8{\cdot}27 \pm 0{\cdot}04$	$8{\cdot}21 \pm 0{\cdot}005$
P_1 (km/s)	$6{\cdot}25 \pm 0{\cdot}02$	$6{\cdot}09 \pm 0{\cdot}03$	$6{\cdot}03 \pm 0{\cdot}09$
S_n (km/s)	$4{\cdot}85 \pm 0{\cdot}10$	$4{\cdot}83 \pm 0{\cdot}02$	$4{\cdot}75 \pm 0{\cdot}01$
S_1 (km/s)	$3{\cdot}54 \pm 0{\cdot}02$	$3{\cdot}68 \pm 0{\cdot}02$	$3{\cdot}55 \pm 0{\cdot}04$
Depth to Mohorovičić discontinuity (km)	36	36	36

It would seem that the old continental shields are seismically very similar.

7. Air waves

The accuracy of sound velocity measurements over a long distance in air is seriously affected by variations in temperature, wind velocity and humidity. From experiments over distances up to 14 km, E. Esclangon (1919) gives $339{\cdot}8 \pm 0{\cdot}1$ m/s for the velocity of sound in dry still air at 15°C. Lately, some observations of air waves recorded from atomic tests have been published. Gutenberg (1946) lists the velocities of direct air waves recorded at five seismological stations, having $4° < \Delta < 10°$, from the New Mexico test. The mean velocity is 350 ± 4 m/s. He gives no similar figures for the Nevada test (Gutenberg, 1952) but states that many records indicated a phase with a velocity of slightly over 300 m/s.

Doyle (1954) recorded air waves from the 1953 atomic explosions in South Australia. At 18 km a velocity of 340 ± 4 m/s was obtained. No temperature corrections are available.

Air waves from the 1954 hydrogen bomb explosions, recorded in Shionomisaki, Japan, at about 36°, have been studied by Yamamoto (1955). In a rediscussion Burke-Gaffney & Bullen (1957) give a mean velocity of 321 m/s with an uncertainty of 5 m/s.

In the Maralinga tests a direct air wave was recorded at 11 Mile Camp from explosion C. The recording is extremely complex but the first onset can be read within 0·1 s. The travel-time is 151·3 s and the corresponding velocity 344 m/s. The uncertainties for explosion C (discussed in Section 4), give an uncertainty of about 6 m/s. Since the various meteorological conditions are not known, the result is in reasonable agreement with previous observations.

Acknowledgments

This work was initiated by Sir Edward Bullard, who arranged for the loan of seismographs from the Scripps Institute of Oceanography, the Bernard Price Institute, and the Pakistan Meteorological Service. The Royal Society provided a grant towards transport of the instruments.

Acknowledgment is also due to the co-operation of the Atomic Weapons Research Establishment, the Australian Department of Supply, the Commonwealth Railways, and to the P.M.G.'s Department in South Australia which made available temporary housing and supplied time signals to the Mt. Bonython site.

The members of the field party have been named above but special mention should be made of Mr R. Green who obtained three valuable records.

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The Motion of the Surface of a Uniform Elastic Half-Space Produced by a Buried Torque-Pulse*

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Summary

In this investigation we determine the motion of the surface of a uniform elastic half-space produced by the application of a torque-pulse at a point below the surface. The axis of the torque is taken to be parallel to the surface and its time variation is assumed to be represented by the Heaviside unit function **H**(t). Our results are compared with those of Pinney (1954), who treated the same problem by a different method. The principal feature of physical interest which we found is that, for ranges where the direct *S* wave is preceded by a diffracted *SP* wave, the displacement at the surface starts with an *infinite* amplitude at the time of arrival of the *SP* wave, and that this is followed by an even stronger *infinity* at the time of arrival of the *S* wave. Also, for small ranges, for which there is no *SP* wave, the displacement starts with a sharp pulse in the form of a Dirac delta function. None of these features was brought out by Pinney's curves. The results are shown in Figures 2, 3 and 4.

1. Introduction

In recent investigations on the problem of the propagation of a pulse of the form of a Heaviside unit function **H**(t) in doubly refracting media, the following result of physical interest was obtained. When the slow-speed wave is incident on an interface at an angle greater than the angle for total reflection, it is partially transformed by diffraction into a high-speed wave, which is propagated along the boundary before re-entering the medium. This high-speed diffracted wave arrives ahead of the direct slow-speed wave, even though its trajectory is longer. The exact solution of the wave equation for this pulse-problem shows that the amplitude becomes *logarithmically infinite* at the time of arrival of the reflected slow-speed wave. On the other hand, no such infinity exists for shorter ranges from the source, which are inside the critical distance for total reflection. This was found to hold for electromagnetic (Pekeris & Alterman 1957), acoustic (Arons & Yennie 1950, Pekeris & Longman 1958, Jeffreys & Lapwood 1957) and seismic (Friedlander 1948, Pekeris & Lifson 1957) pulses. In the acoustic case some experimental evidence in support of this theoretical result was produced by Arons & Yennie (1950). It was also suggested (Pekeris & Longman 1958) that this effect may have a bearing on the observations of Tatel & Tuve (1955) on the behaviour of the

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reflection from the Mohorovičič discontinuity in the Earth's crust as one passes the critical range.

In this investigation we wish to determine the motion of the surface of a uniform elastic half-space produced by the application of a torque-pulse at a point situated at a depth H below the surface. The torque is taken to be oriented parallel to the surface, with its axis pointing in the positive x-direction, as is shown in Figure 1. The time variation of the torque is assumed to be represented by the Heaviside unit function $\mathbf{H}(t)$. The source emits shear waves (S) only. On reaching the surface, the incident S wave is reflected partially as an S wave and partially as a P wave, if the angle of incidence is less than the angle for total reflection. For ranges beyond the angle of total reflection, we have from geometrical optics that all of the incident energy is reflected as an S wave. However, because of the sphericity of the incident wave front, a *diffracted P* wave is propagated along the surface. We shall designate this diffracted wave as the SP wave. Under these circumstances a point on the surface first receives the SP wave and then the direct S wave, although the path of the former is longer.

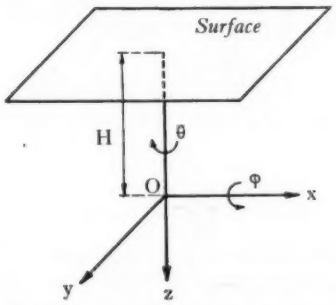

FIG. 1.—A torque-source, whose axis is oriented along the x-axis is applied at the point O below the surface. θ is the angle in the x-y plane; ϕ is the angle in the y-z plane.

In a previous investigation (Pekeris & Lifson 1957) of the case of a buried seismic point source of the form of a *downward force* varying like $\mathbf{H}(t)$, it was found that at the time of arrival of the S wave the amplitude becomes logarithmically infinite. For the problem at hand of the buried torque-source, one would expect an even stronger infinity for the displacement at the surface. Now the buried torque-source problem was treated previously by Pinney (1954), and his computed curves for the displacement of the surface do not show any infinities. We have therefore undertaken to resolve this problem by a different method, and to compare the results with Pinney's solution. It turns out that both solutions agree and give strong infinities at the epoch referred to, as well as at other times. The reason why Pinney missed these singularities, which are the features of the solution of principal interest to the seismologist, is that his method of numerical evaluation of the integrals appearing in his solution becomes inaccurate just where the singularities occur.

2. Formal solution

We adopt the system of coordinates shown in Figure 1, with the x-axis

pointing in the direction of the axis of the applied torque. Let the displacement vector $\mathbf{u}$ be given by

$$\mathbf{u} = \operatorname{grad}\phi + \operatorname{curl}\mathbf{T}, \tag{1}$$

then the boundary conditions can be satisfied by taking

$$T_x = Q, \qquad T_y = 0, \qquad T_z = \partial P/\partial x, \qquad \phi = \partial\psi/\partial y. \tag{2}$$

The components of $\mathbf{u}$, and the dilatation Δ are then given by

$$u = \psi_{xy} + P_{xy}, \tag{3}$$

$$v = \psi_{yy} + Q_z - P_{xx}, \tag{4}$$

$$w = \psi_{yz} - Q_y, \tag{5}$$

$$\Delta = \operatorname{div}\mathbf{u} = \nabla^2\psi_y, \tag{6}$$

where the subscripts denote partial differentiation. On substituting into the equations of motion, we find that they are satisfied if

$$\nabla^2 Q - k^2 Q = 0, \tag{7}$$

$$\nabla^2 P - k^2 P = 0, \quad \nabla^2\psi - h^2\psi = 0, \tag{8}$$

where

$$k^2 = p^2/c^2, \quad h^2 = p^2/c_p^2, \quad c^2 = \mu/\rho, \quad c_p^2 = (\lambda+2\mu)/\rho = 3c^2. \tag{9}$$

Here c denotes the velocity of shear (S) waves, c_p the velocity of compressional (P) waves, and p denotes the operator $\partial/\partial t$. λ and μ denote the Lamé elastic constants of the medium, which are assumed to be equal.

Consider now the solution of the wave equation (7):

$$Q = R^{-1} f[t-(R/c)], \quad R = (x^2+y^2+z^2)^{\frac{1}{2}}, \tag{10}$$

where f is an arbitrary function.

Taking a cylindrical system of coordinates (r, ϕ, x), as shown in Figure 1, with $r = (y^2+z^2)^{\frac{1}{2}}$, we find from (1) and (2) that Q represents a torsional wave with only one component of displacement u_ϕ along circles in planes normal to the x-axis:

$$u_\phi = -\frac{\partial Q}{\partial r}, \qquad \tau_{\phi x} = \mu\frac{\partial u_\phi}{\partial x} = -\frac{\partial^2 Q}{\partial r\,\partial x} = -\frac{\mu r x}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial Q}{\partial R}\right). \tag{11}$$

Let T' denote the torque which the matter to the left of a plane normal to the x-axis exerts on the matter to the right of this plane. We have

$$T' = -2\pi\int_0^\infty \tau_{\phi x} r^2\,dr = -2\pi\mu x\left[(R^2-x^2)\left(\frac{f}{R^3}+\frac{\dot f}{cR^2}\right)+\frac{2f}{R}\right]_{R=x}^{R=\infty}$$

$$= 2\pi\mu x\left\{-\frac{\dot f(-\infty)}{c}+\frac{2}{x}f[t-(x/c)]\right\}. \tag{12}$$

In the limit of vanishing x we get

$$T' \to 4\pi\mu f(t). \tag{13}$$

The same torque T' is exerted at O by the matter on the right of the yz plane on the matter to the left, so that the total torque T exerted by the source represented by Q in (10), is

$$T = 8\pi\mu f(t). \tag{14}$$

The operational form of the solution of the wave equation (7) for a torque-source of magnitude T and varying with time like $\mathbf{H}(t)$ is therefore given by

$$Q_0 = \frac{T}{8\pi\mu R} e^{-kR}. \tag{15}$$

One of the boundary conditions is the vanishing of all the components of stress on the surface, which is a horizontal plane:

$$\tau_{xz} = \tau_{yz} = \tau_{zz} = 0, \qquad z = -H. \tag{16}$$

Accordingly we use the representation

$$Q_0 = \frac{T}{8\pi\mu R} e^{-kR} = \frac{T}{8\pi\mu} \int_0^\infty \frac{\xi}{k\beta} e^{-k\beta|z|} J_0(\xi r)\, d\xi, \tag{17}$$

with

$$r = (x^2+y^2)^{\frac{1}{2}}, \qquad k\beta = (\xi^2+k^2)^{\frac{1}{2}}, \qquad k\alpha = [\xi^2+(k^2/3)]^{\frac{1}{2}}. \tag{18}$$

Following the method of Lamb (1904), we represent the solution of the wave equations (7) and (8) by ξ-integrals like (17), and, denoting the integrands by bars, take

$$\bar{\psi} = Ae^{-k\alpha z} J_0(\xi r), \qquad \bar{P} = Be^{-k\beta z} J_0(\xi r), \tag{19}$$

$$\bar{Q} = (T/8\pi\mu)(\xi/k\beta) e^{-k\beta|z|} J_0(\xi r) + Ce^{-k\beta z} J_0(\xi r). \tag{20}$$

The three boundary conditions (16) yield

$$A = (T/8\pi\mu M) 4\xi(2\xi^2+k^2) e^{-k(\alpha+\beta)H}, \tag{21}$$

$$B = -(T/8\pi\mu M) 8\xi k^2 \alpha\beta e^{-2k\beta H}, \tag{22}$$

$$C = -(T/8\pi\mu M)(\xi/k\beta)[(2\xi^2+k^2)^2 + 4\xi^2 k^2 \alpha\beta] e^{-2k\beta H}, \tag{23}$$

with

$$M = (2\xi^2+k^2)^2 - 4\xi^2 k^2 \alpha\beta. \tag{24}$$

Substituting in (5), and putting $\xi = kx$, we get for the vertical displacement w_0 at the surface

$$w_0 = \frac{Tk^2 \sin\theta}{2\pi\mu} \int_0^\infty \frac{x^2[x^2+(\frac{1}{3})]^{\frac{1}{2}} \exp[-kH(x^2+1)^{\frac{1}{2}}] J_1(krx)\, dx}{\{(2x^2+1)^2 - 4x^2(x^2+1)^{\frac{1}{2}}[x^2+(\frac{1}{3})]^{\frac{1}{2}}\}}, \tag{25}$$

where θ is measured from the direction of the x-axis and $\sin\theta = y/r$ (see Figure 1). Similar expressions can be derived for the other components of displacement. In the sequel we confine our attention to w_0 only.

3. Operational interpretation

The operational interpretation of (25) can be readily carried out by the method developed by the senior writer (Pekeris 1956; see also Cagniard 1939, Pinney 1954). It can be expresed in the form

$$w_0 = (T/\pi^2\mu Rr)\sin\theta(\partial S/\partial\tau), \tag{26}$$

where

$$\tau = (ct/R), \quad \alpha = [x^2+(\tfrac{1}{3})]^{\frac{1}{2}}, \quad \beta = (x^2+1)^{\frac{1}{2}}, \quad f(x) = \frac{\alpha}{[(2x^2+1)^2-4x^2\alpha\beta]}, \tag{27}$$

and S is defined below. The operational interpretation of (25) differs in the two cases when the horizontal range r on the surface is less than or greater than the critical distance $H/\sqrt{2}$ for total reflection. In the latter case the motion becomes more complex due to the appearance of the diffracted SP wave.

We have for $r < H/\sqrt{2}$:

$$S = 0, \qquad \tau < 1, \tag{28}$$

$$S = S_1 = \mathrm{Re}\int_0^{\frac{1}{2}\pi} \beta(\tau-h\beta)f(x)\,d\phi, \qquad \tau > 1. \tag{29}$$

Here Re denotes the real part, and

$$\beta = h\tau + i\mu_0\sin\phi, \qquad \mu_0 = [(1-h^2)(\tau^2-1)]^{\frac{1}{2}}, \tag{30}$$

$$h = (H/R), \qquad x = (\beta^2-1)^{\frac{1}{2}}, \qquad \alpha = [\beta^2-(\tfrac{2}{3})]^{\frac{1}{2}}. \tag{31}$$

When $r > H/\sqrt{2}$, the first arrival is the diffracted SP wave, at a time corresponding to

$$\tau^* = (\tfrac{2}{3})^{\frac{1}{2}}h + (\tfrac{1}{3})^{\frac{1}{2}}(1-h^2)^{\frac{1}{2}}, \tag{32}$$

while the direct S wave arrives later at $\tau = 1$. We have then

$$S = 0, \qquad \tau < \tau^*, \tag{33}$$

$$S = -S_2 = -\,\mathrm{Im}\int_0^{\frac{1}{2}\pi} \beta(\tau-h\beta)f(x)[(\kappa^2+\sin^2\phi)^{-\frac{1}{2}}]\cos\phi\,d\phi, \quad \tau^* < \tau < 1. \tag{34}$$

Here Im denotes the imaginary part, and

$$\delta = [(\tfrac{2}{3})^{\frac{1}{2}}-h\tau], \qquad m = (\delta^2-\kappa^2m^2)^{\frac{1}{2}} \qquad \kappa m = [(1-h^2)(1-\tau^2)]^{\frac{1}{2}}, \tag{35}$$

$$\beta = h\tau + m(\kappa^2+\sin^2\phi)^{\frac{1}{2}}, \quad x = i(1-\beta^2)^{\frac{1}{2}}, \quad \alpha = i[(\tfrac{2}{3})-\beta^2]^{\frac{1}{2}}. \tag{36}$$

After the arrival of the direct S wave, we get

$$S = S_1 - S_3, \qquad 1 < \tau < (\tfrac{2}{3})^{\frac{1}{2}}/h, \tag{37}$$

$$S = S_1, \qquad \tau > (\tfrac{2}{3})^{\frac{1}{2}}/h, \tag{38}$$

where

$$S_3 = \delta I \int_0^{\frac{1}{2}\pi} \beta(\tau - h\beta) f(x)[(\mu_0^2 + \delta^2 \sin^2\psi)^{-\frac{1}{2}} \cos\psi]\, d\psi, \tag{39}$$

$$\beta = h\tau + \delta \sin\psi, \qquad x = i(1-\beta^2)^{\frac{1}{2}}, \qquad \alpha = i[(\tfrac{2}{3})-\beta^2]^{\frac{1}{2}}. \tag{40}$$

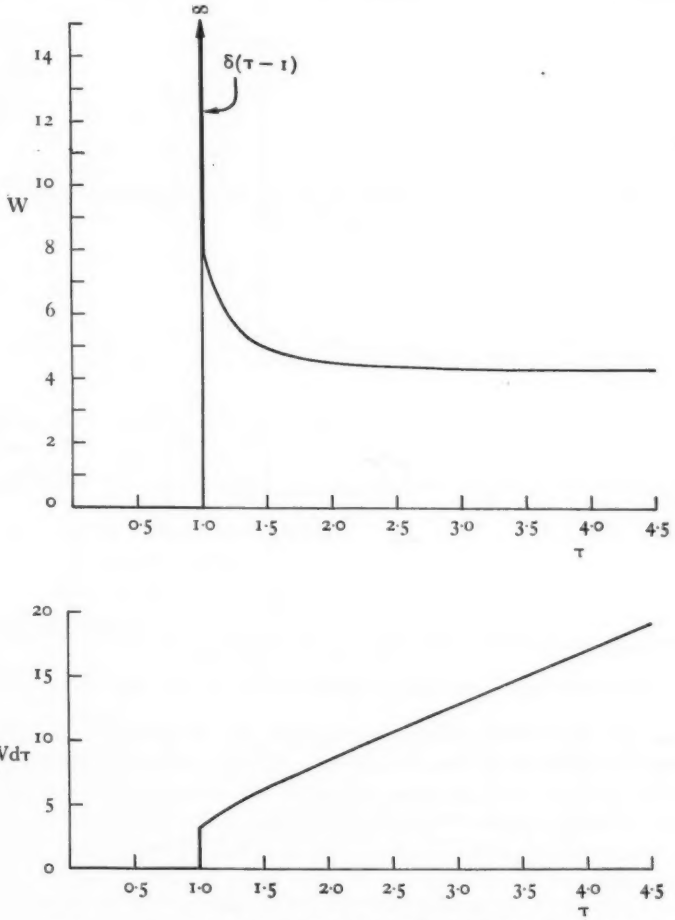

FIG. 2.—Vertical displacement at the surface w_0 as a function of time.
$W = (16\pi^2 H^2/T \sin\theta) w_0$, $\tau = (ct/R)$, $R = (x^2+y^2+z^2)^{\frac{1}{2}}$.

4. Discussion of results

We have evaluated the quantity W, which is related to the vertical component of displacement at the surface w_0 by

$$W = \frac{16\pi^2 \mu H^2}{T \sin\theta} w_0 = \frac{16H^2}{R\tau} \frac{\partial S}{\partial \tau}. \tag{41}$$

W has also been computed by Pinney (1954) by his method*, thus allowing direct comparison of results. The integrals S_1, S_2 and S_3 in (29), (34) and (39) were computed on WEIZAC using a 48 point Gaussian quadrature formula (Davis & Rabinowitz 1956). In order to determine S at a given value of the reduced time $\tau = (ct/R)$, one or two (as in Equation 37) of these integrals had to be evaluated. $\partial S/\partial\tau$ was then determined from $[S(\tau+\epsilon)-S(\tau-\epsilon)]/2\epsilon$, using values of 0·001 and 0·0001 for ϵ, and the extent to which these results agreed was taken to indicate their accuracy.

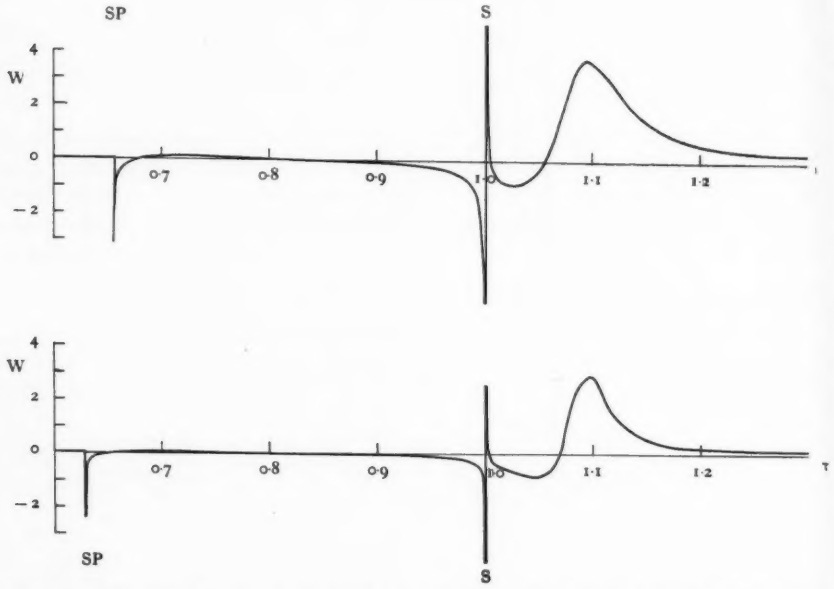

FIG. 3.—Vertical displacement at the surface w_0 as a function of time. $W = (16\pi^3H^2/T\sin\theta_0)w_0$, $\tau = (ct/R)$, $R = (x^2+y^2+z^2)^{\frac{1}{2}}$.
SP denotes the diffracted wave; *S* the shear wave. $A: r = 10H$; $B: r = 16H$.

Figure 2A shows W as a function of τ for the range $r/H = 0{\cdot}25$ for which the diffracted *SP* wave does not yet exist. The quantity $(16H^2/Rr)S(\tau)$, whose τ-derivative is equal to W, is shown in Figure 2B. It is seen that the latter curve starts with a *finite jump* at $\tau = 1$, *which contributes a* $\delta(t)$ (Sherwood 1958) *function to* W at this epoch. This $\delta(t)$ pulse is not shown in Pinney's curve. Otherwise the results agree.

Figure 3 shows W as a function of τ for the ranges $r = 10H$ and $r = 16H$, for which the diffracted *SP* wave exists. The *SP wave starts with an infinite amplitude*. The singularity is of the form $-(\tau-\tau^*)^{-\frac{1}{2}}$. After this sharp arrival, the displacement remains small until the arrival of the S wave at $\tau = 1$, when it *becomes infinite like* $(\tau-1)^{-1}$. Neither of these singularities is shown in Pinney's curves.

Figure 4 shows an enlargement of the curves in Figure 3 near the time of arrival of the S-wave at $\tau = 1$. The triangles and the dashed portions of the curves represent Pinney's results.

* Pinney's τ is equal to our τ multiplied by R/H.

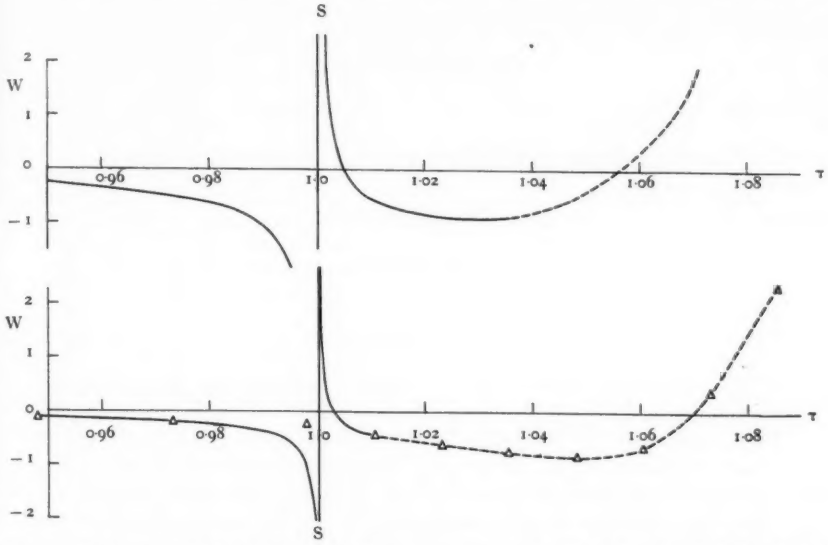

FIG. 4.—Enlargement of the curves in Figure 3 near $\tau = 1$. The triangles and the dashed portions of the curves represent Pinney's results. $A : r = 10H$; $B : r = 16H$.

We have also recomputed W using Pinney's formulae*, and the results obtained are in complete agreement with the values derived by our method, except for the delta function referred to above.

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* Equations 4.16 and 4.17 of Pinney's paper contain misprints. The upper limits of integration should be K^2 and not κ^2.

The Times of P up to 30° (Second Paper)*

Harold Jeffreys

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Summary

A new set of earthquakes is studied to check the times of *P* in European earthquakes up to 30°, and especially the behaviour near 20°. The corrections to the author's table of 1939 agree closely with those found from an earlier set in 1935. Lehmann's observations of *Pd* are consistent with her interpretation if *Pr* overtakes *Pd* at about 14°, but in any case $dt/d\Delta$ for *Pd* is remarkably constant. It seems probable that there is triplication and that the observations refer to a neighbourhood of the more distant cusp on the time curve.

No useful new data were available for the Japanese region.

Several earthquakes in North and Central America showed that from 8° to 30° the 1939 table is correct to a fraction of a second apart from a possible additive constant. Additional information on the times at distances under 8° was given by a paper of Lehmann. The data in this indicate that the European velocity is nearly correct for North America, but that, relative to 5°, the times from 8° to 30° for American earthquakes are about 2 s shorter than for European ones.

1. European earthquakes

In two former papers (Jeffreys 1952, 1953) I examined new sets of earthquakes with "European", Japanese and American epicentres. Corrections to the Jeffreys–Bullen *P* table (Jeffreys 1939) were found from European and Japanese epicentres; the results from American ones were unsatisfactory, partly through internal discrepancies and partly on account of the difficulty of separating corrections to the table from errors of the epicentres. Substantial corrections, mainly owing to a negative correction to $dt/d\Delta$ up to 8°, were found for Europe; but those in the specially interesting region about 20° were still doubtful.

In this paper I study a new selection of European earthquakes from the I.S.S. These were chosen because all had observations about 20°. As before, the observations were classified by quadrants before analysis, to reduce the effects of errors in the epicentres. The useful quadrants are indicated in the third column.

Precautions as in my papers of 1952 and 1953 were taken to avoid errors due to focal depth and multiplicity.

The observations were grouped by ranges: 2°–8°, 8°–12°, 12°–15°, 15°–17°·5, 18°, 19°, 20°, 20°·5–25°, 25°–30°. Corrections were expected to vary little in each

* The first paper on this subject appeared in *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **6**, 348, 1952.

Table 1

List of earthquakes used

Date	Useful Quadrants	Epicentre	Date	Useful Quadrants	Epicentre
1946 Feb. 12	N, E	Spain	1948 Feb. 9	N, E, W	Levant
Feb. 21	N, W	Levant	Mar. 6	N	Levant
Apr. 5	N, E, W	Levant	June 30	N, E, W	Italy
May 11	S	N. Atlantic	July 8	S	N. Atlantic
May 30	N, E, S	Switzerland	July 24	N, W	Levant
July 16	N, E, W	Levant	Aug. 27	N, W	Jugoslavia
Oct. 13	N, W	Levant			
			1949 Jan. 14	N, W	Turkey
1947 May 11	N, W	Italy	Feb. 5	N, W	Turkey
June 4	N, W	Levant	Mar. 2	S	N. Atlantic
Aug. 30	N, E, W	Levant	May 13	W	Turkey
Oct. 3	N	Levant	July 8	S	N. Atlantic
Oct. 6	N	Greece	Aug. 17	N, E, W	Caucasus

range except the first, and the I.S.S. residuals (against the 1939 table) were used directly. For the first range the observations were grouped at intervals of 1° and compared with my previous table (1953, p. 563). This introduced a discontinuity into the standard of comparison, but this could be treated later. The data gave standard error 1·84 s on 415 degrees of freedom. Comparisons between distances by the method used previously gave the corrections in Table 2 (in order of estimation), the range 15°–17°·5 being taken as standard.

Table 2

Range deg	Correction s	Weight	χ^2	d.f.	Entries rejected
15–17·5	0				
20·5–25	−0·1	44	30·9	28	0
25–30	+0·2	51	34·2	31	0
12–15	−0·5	53	34·6	29	0
8–12	+0·5	52	28·6	30	5
2–8	+2·5	42	30·8	25	1
18	+0·1	33	23·2	20	1
19	−0·6	42	16·1	23	0
20	−0·4	30	13·9	21	3
		Totals	212·3	207	

The total χ^2 is normal, but has been obtained as a result of rejecting some entries. (An entry is a mean residual over a range of distance for one earthquake.) The explanation is partly that the law of error is not normal, and in estimating the means and standard error this was dealt with by rejecting residuals over 5 s. If the law was normal with $\sigma = 2$ s this would make little difference, and previous trials have shown that with the actual law it will not habitually underestimate σ by more than 10 per cent or so. Several of the rejected entries depended on a single discrepant observation, which would have been rejected at an earlier stage if there had been a standard of comparison. What may be more serious is that the method compares the mean residual at each stage with the whole of those at earlier stages, as if these were exactly known. For instance, if the difference between

15°–17°·5 and 20°–25° was altered by its standard error, this would alter the value for 25°–30° obtained by comparison with the two together. For this reason the weights at this and later stages are too high and might be reduced by 30 per cent or so. The correct treatment would of course be to set up a least squares solution for eight differences of corrections, but this has not been thought worth while. The values for 18°, 19° and 20° are all in comparison with the same standard and should be comparable between themselves.

At 20° one rejected entry depended on a single observation. The others rejected were as follows:

		Obs. at 20°	Weight	Final mean residual	χ^2	Crude residuals
				s		s
1946 Oct. 13	N	5	3·4	−3·0	9	−3 −4 −4 −2 −2
1948 July 24	W	4	2·7	+4·1	14	0 +1 +2 +1

The readings are very consistent in both cases, but the means flatly contradict each other. In my earlier treatment (1952, p. 354) a mean residual of −1·4s, weight 16, was found at this distance, and I remarked that omission of a single series (1938 April 19, W) would have changed this to −0·1 s. This was what led to the doubt at this distance. There was then one discrepant entry; here there are two in opposite directions. It is perhaps not out of the question that with

Table 3

Mean residuals

Earthquake	Quadrant	Mean Residual	Weight	Earthquake	Quadrant	Mean Residual	Weight
		s				s	
1946 Feb. 12	N	−0·9	15	1948 Feb. 9	N	+0·7	20
	E	+0·9	2		E	−0·4	2
Feb. 21	N	−0·5	12		W	−1·1	14
	W	+0·6	13	Mar. 6	N	−0·6	12
Mar. 17	N	+1·1	4	June 30	N	−1·2	25
Apr. 5	N	+0·2	16		E	+0·6	6
	E	+1·7	6		W	−1·0	8
	W	+2·1	5	July 8	S	−0·7	13
May 11	S	−0·5	9	July 24	N	−0·8	25
May 30	N	−3·0	12		W	−2·7	8
	E	−0·1	7	Aug. 27	N	−0·5	13
	S	+1·4	6		W	+0·9	9
July 16	N	+0·1	17				
	E	−1·6	3	1949 Jan. 14	N	−0·8	8
	W	+2·6	8		W	−0·9	14
Oct. 13	N	+0·4	11	Feb. 5	N	+1·2	8
	W	−4·7	7		W	−2·4	7
1947 May 11	N	+0·8	20	Mar. 2	S	−1·2	17
	W	−0·5	15	May 13	W	−1·7	20
June 4	N	−2·0	17	June 17	S	+1·1	17
	W	−0·6	16	July 8	S	+1·2	15
Aug. 30	N	−0·7	25	July 10	W	+0·8	6
	E	+3·1	4	Aug. 17	N	−1·4	8
	W	+3·8	6		E	−1·2	11
Oct. 3	N	+0·4	10		W	−0·2	21
Oct. 6	N	−1·4	22				

a departure from the normal law there might have been a few cases in over 200 where observations departing seriously from the true value should agree among themselves. Whether this is the explanation of the anomaly or not, we have 18 consistent entries, and can see that these two are in some way exceptional.

The mean residuals after allowance for the corrections are as follows. They may be useful if the epicentres are recalculated.

To make the results comparable with my 1953 ones, which are adapted to a surface focus, 2·5 s must be subtracted.

Table 4

Comparison of present and 1953 results

Δ deg.	Present Correction s	Weight	Δ deg.	1953 Correction s	Weight	χ^2	d.f.
2–8	0·0		2–8	0·0			
8–12	−2·0	52	8–12·5	−3·7	9	12	20
12–15	−3·0	53	12·5–17·5	−3·6	(20)	25	
15–17·5	−2·5	(42)	15–19	−2·8	52	49	41
18	−2·4	33	19–20	−3·9	30	28	24
19	−3·1	42	20–25	−2·2	64	43	25
20	−2·9	30	25–30	−2·9	31	20	13
20·5–25	−2·6	44					
25–30	−2·3	51					

The bracketed weights are for the comparison of the standard distance with 2°–8°.

Table 5

Smoothed times

Δ	Present Time min	s	δ s	δ^2	1953 Time min	s	δ s	δ^2
14	3	19·0			3	19·0		
			13·3				13·7	
15		32·3		−0·3		32·7		−0·9
			13·0				12·8	
16		45·3		−0·4		45·5		−0·4
			12·6				12·4	
17		57·9		−0·3		57·9		−0·3
			12·3				12·1	
18	4	10·2		−0·3	4	10·0		−0·3
			12·0				11·8	
19		22·2		−0·6		21·8		−0·4
			11·4				11·4	
20		33·6		−0·5		33·2		−0·5
			10·9				10·9	
21		44·5		−0·6		44·1		−0·3
			10·3				10·6	
22		54·8		−0·2		54·7		−0·4
			10·1				10·2	
23	5	4·9		−0·2	5	4·9		−0·2
			9·9				10·0	
24		14·8		−0·1		14·9		−0·4
			9·8				9·6	
25		24·6				24·5		

The contributions to χ^2 show no inconsistency, though the data were combined in different orders and somewhat different ranges of distance. The principal difference is about 20°, −0·9 s, weight 15, corresponding to a standard error of about 0·5 s and it is not remarkable for an extreme value. The present treatment goes a little more into detail than the previous one (but some of the information was recovered in the previous one by a reclassification of the residuals at shorter intervals). The previous corrections have been smoothed (Table V, p. 563) and the only change worth making is to increase the time at 20° by 0·5 s. If we make this change a permissible smoothing gives the times in Table 5.

The differences in the times are trivial in relation to the uncertainties (±0·3 s to 0·5 s); but the 1953 times give a considerable change of slope at 15° and no other anomaly, while the present ones give a curvature beginning at 14°, increasing slightly to 18°–21° and then abruptly decreasing. The observational material is slightly in favour of the present solution, and in either case the curvature from 14° to 25° is substantially greater than at other distances.

If Lehmann's (1934) interpretation of a second arrival (*Pd*) at about 20° is correct, the times should connect with those at 14° by a curve with a downward curvature. I take her distances CII for the Azores earthquake (1931 May 20): these use the epicentre 37°·6 N, 16°·0 W. The earthquake was used in my discussion with epicentre 37°·4 N, 15°·9 W, taken from the I.S.S. In a revision (not published) I got 37°·44 ± 0°·08 N, 16°·17 ± 0°·05 W. To have a comparison as nearly independent of the epicentre as possible I take as trial times of *Pd* the formula

$$t = 3\text{ min } 19{\cdot}0\text{ s} + 13{\cdot}70(\Delta - 14^\circ)$$

obtained by linear extrapolation from 14°, and for *Pr* those just found. The theoretical interval is then compared directly with the observed one. Since the difference of $dt/d\Delta$ for *Pr* and *Pd* is much less than either separately, the intervals are much less affected by any error of the epicentre than the total times. Lehmann's distances are used. Thus the observed times are earlier than the extrapolated ones and permit a small downward curvature for *Pd*. This was not true for comparison with the 1939 table, which made *Pr* overtake *Pd* at about 19°·6, and made all the *Pd* observations lie too late to give a downward curvature. The stations are in azimuths near 45°, and the corrected epicentre would increase the distances by 0°·19 ± 0°·07. The difference of slope would make the calculated

Table 6

Observed and calculated differences Pd–Pr

Station	Δ	*Pd–Pr* calc.	*Pd–Pr* obs.	O − C
	deg	s	s	s
Strasbourg	20·4	+ 8·7	+5·4	−3·3
Zurich	20·5	+ 9·0	+4	−5
De Bilt	20·7	+ 9·6	+5	−4·6
Chur	20·9	+10·0	+7	−3
Karlsruhe	21·0	+10·4	+5·5	−4·9
Stuttgart	21·3	+11·4	+8	−3·4
Ravensburg	21·3	+11·4	+8	−3·4
			Mean:	−3·9 ± 0·3

difference about 0·6 s larger in magnitude. If the curvature contributes −4 s or so at 21° and is due to a term in Δ^3, its effects up to 14°. would hardly be detectable. But as the amplitudes of *Pd* are larger than those of *Pr*, which is itself specially large at this distance, it is probable that a considerable curvature for *Pd* begins quite close to 20°. But $dt/d\Delta$ for any part of *Pd* cannot be less than that for any part of *Pr*, and this from 15° to 16° is already at least 12·8 s/deg. It seems that the existence of *Pd* must depend on a rather delicate adjustment of conditions, and it would not be at all surprising if it is present for earthquakes in some regions and not in others. Lehmann's reproduced records are convincing evidence that there was something to read, whatever the interpretation may be.

Beyond Ravensburg and up to Copenhagen at 26°·3 Lehmann had 13 stations, at none of which was *Pd* read. It seems that the most probable interpretation is that the velocity of *P* varies continuously with depth; that it first decreases slightly to account for the constancy of $dt/d\Delta$ to 14°; and that it then increases a little, but in a sufficiently short distance to produce triplication. Then Lehmann's data would refer to the more distant cusp on the time curve.

2. Japanese earthquakes

Search was made for earthquakes in the region of Japan with numerous observations about 20°, but none was found that was suitable.

3. North American earthquakes

The 1939 *P* table has not been checked for distances under 30° in America; the data were for European earthquakes supplemented at large distances by observations, mostly in Europe, of earthquakes in other regions. The increase in the number of North American stations now makes a check possible. The earthquakes used are as follows.

1945 Apr. 21, June 30, Oct. 29.
1946 Feb. 15, Mar. 13 (13 h 21 min and 13 h 49 min), May 9, July 18 d 6 h.
1947 Aug. 7 d 0 h.
1948 Feb. 28.
1949 Aug. 22, Oct. 31.

Classification of the readings by quadrants showed few cases where there were many observations within 8°. I assume for each quadrant a correction a; and corrections to the table

8° − 12°	x_1
12° − 15°	x_2
15° − 17°·5	x_3
20°·5 − 25°	x_4
25° − 30°	x_5.

For each quadrant I made normal equations for a and the five corrections and eliminated a. Then the equations for the corrections were combined for all earthquakes and solved. The problem is of course indeterminate, since an addition to a and an equal subtraction from all the x s would alter no residual; and the normal equations are essentially for differences of the x s, and at the last stage of the

solution there are two equations differing only in sign. Linear combinations with independent uncertainties are as follows.

$$\begin{aligned} x_4 - 0{\cdot}03\,x_1 - 0{\cdot}27\,x_2 - 0{\cdot}37\,x_3 - 0{\cdot}34\,x_5 &= +0{\cdot}5 \pm 0{\cdot}4 \\ x_2 - 0{\cdot}29\,x_1 - 0{\cdot}49\,x_3 - 0{\cdot}22\,x_5 &= +0{\cdot}2 \pm 0{\cdot}5 \\ x_3 - 0{\cdot}38\,x_1 - 0{\cdot}62\,x_5 &= +0{\cdot}0 \pm 0{\cdot}6 \\ x_1 - x_5 &= -0{\cdot}2 \pm 1{\cdot}0 \end{aligned}$$

No corrections are worth making in these ranges of distance.

For 18° to 20° I get the following means against the other distances.

deg.		χ^2	d.f.
18	$+0{\cdot}4 \pm 0{\cdot}4$	13·8	12
19	$+0{\cdot}1 \pm 0{\cdot}5$	7·7	10
20	$-0{\cdot}8 \pm 0{\cdot}6$	12·1	5
	$(-1{\cdot}4 \pm 0{\cdot}7)$	3·8	4

The second row of results for 20° comes from rejection of one entry. The general conclusion is that for earthquakes in North America the 1939 table is satisfactory to within a second from 8° to 30° and there is no evidence calling for any change.

For the observations within 8° I fitted corrections $a' + b\Delta$ to the residuals against the 1953 table. The only useful information for b came from 1946 March 13, E and W, which gave respectively

$$b = -0{\cdot}15 \pm 0{\cdot}4\,\text{s/deg}, \qquad b = -0{\cdot}05 \pm 0{\cdot}9\,\text{s/deg},$$

and in all $b = -0{\cdot}1 \pm 0{\cdot}3$ s/deg. This is slight evidence that $dt/d\Delta$ at short distances agrees better with the European than with the Japanese value.

Lehmann (1955) has made a study of P and S in N.E. America. The observations were not classified by quadrants, and linear forms were fitted up to 12° by least squares. Thus the uncertainties may be underestimated, and the question of a curvature before 12° is undecided. I have treated the observations in the same way as the I.S.S. data. There were many more within 8°. Unfortunately there were not often more than two in the same quadrant for any of the ranges of distance used, and a standard error could not be estimated till a general solution had been made.

The observations up to 8° gave standard deviation 1·0s on 11 degrees of freedom, $b = (+0{\cdot}15 \pm 0{\cdot}12)$ s/deg. Combining with the previous data (which were given half weight on account of the larger standard deviation) gave

$$b = (+0{\cdot}11 \pm 0{\cdot}11)\text{s/deg}.$$

This is enough to show that the velocity of P at short distances agrees better with the European than with the Japanese value. To reduce as far as possible the effect of the uncertainty of b the mean residuals up to 8° were adapted to distance 5°.

Between 20° and 30° there were very few observations in quadrants where they could be usefully compared with other distances. The solutions for x_1, x_2, x_3 gave the forms with independent uncertainties:

$$\begin{aligned} x_1 - 0{\cdot}50\,x_2 - 0{\cdot}50\,x_3 &= +2{\cdot}5 \pm 1{\cdot}2 \\ x_2 - x_3 &= +1{\cdot}2 \pm 1{\cdot}3 \end{aligned}$$

with standard deviation 1·8s on 8 degrees of freedom.

Because of the much smaller amount of material the uncertainties are much

larger than those of the I.S.S. set. It is possible, however, to adopt values from the previous set:

$$x_1 = 0, \quad x_2 = +0{\cdot}2, \quad x_3 = +0{\cdot}1, \quad x_4 = +0{\cdot}6, \quad x_5 = +0{\cdot}2,$$

and compare the mean residual against these with the mean residual already estimated at 5°. Comparison was possible for four quadrants.

			5° s	weight	$a+x_1$ s	weight
1935	Nov. 1	S	−0·2	3	−4·9	2
1939	Oct. 19	W	+0·9	2	−1·5	2
1940	Dec. 20	W	+0·4	4	−6·3	3
1952	Apr. 9	E	+0·2	3	−6·3	6

The weighted mean difference is $-5{\cdot}4\,\text{s} \pm 0{\cdot}6\,\text{s}$; $\chi^2 = 4{\cdot}1$ on 3 d.f. This would be (within 1 s or so) the correction to the 1939 table for distances from 8°·5 to 30°. The corrections for Europe in this range are between −2 s and −3 s, so it looks as if (relative to 5°) the American times are shorter by about 3 s. However, Lehmann's earthquakes were in the eastern half of North America. The I.S.S. series were mostly near the west coast, and a possible alternative would be a variation within North America.

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Rock Creep and Thermal Instability

Harold Jeffreys

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Summary

The theory of instability of a fluid heated below can be adapted without difficulty to a rock satisfying the modified Lomnitz law of creep. It is found that instability is impossible and therefore that steady convection currents would not arise.

The theory of instability of a fluid heated below assumes a finite viscosity, and it is desirable to examine whether it needs serious change for rocks obeying Lomnitz's law of creep or my modification (Jeffreys 1958). It turns out that the solution for finite viscosity can be adapted with surprising ease.

In the standard case of viscosity, the temperature disturbance V' at marginal stability satisfies an equation of the form (Jeffreys & Bland 1950):

$$\nabla^6 V' = \frac{G}{\nu}\nabla_1^2 V', \tag{1}$$

where G depends on various parameters of the problem, including the excess temperature gradient over the adiatabic; ν, the kinematic viscosity, is made explicit.

Now for a viscous liquid the shear stress is $\rho\nu$ times the rate of shear; that is, if v is the displacement, the shear stress contains

$$\rho\nu \frac{\partial}{\partial t}\frac{\partial v_k}{\partial x_i}.$$

For creep, the initial behaviour is similar to that for viscosity, but this stage lasts only for a small fraction of a second. At greater times the rate of strain decreases, and the stress contains

$$\mu(1+kp^{-\alpha})^{-1}\,\partial v_k/\partial x_i, \tag{2}$$

with $k > 0, 0 \leqslant \alpha < 1$, where p is the Heaviside operator. Hence we get a formal adaptation of (1) by replacing ν by an operator of the form $(\mu/\rho p)(1+kp^{-\alpha})^{-1}$. For $\alpha = 1$ this would express elasticoviscosity; and for general α (1) will be replaced by

$$\nabla^6 V' = Hp(1+kp^{-\alpha})\nabla_1^2 V', \tag{3}$$

with $H = G\rho/\mu$. For $\alpha = 1$ the lowest power of p reproduces (1).

(1) was found by neglecting all powers of p but the lowest, and hence, since only integral powers of p arose, (3) will be correct to the *two* lowest powers of p.

Then the lowest power on the right leads to

$$\nabla^6 V' = Hkp^{1-\alpha}\nabla_1^2 V', \quad (4)$$

and the critical case of a steady motion ($p = 0$) implies

$$\nabla^6 V' = 0. \quad (5)$$

But subject to the boundary conditions of the problem this has no solutions at all; for this would be the condition that marginal stability should occur at zero temperature gradient.

Hence under the law of rock creep, heating below cannot produce convection currents.

Physically the reason is as follows. Consider steady convection in the viscous case. The excess temperature in the heat columns is maintained against conduction by warm liquid flowing in along the bottom, and the velocity is produced by shear stresses. The essential point is that constant shear stress in this case implies constant rate of shear. But in the case of creep, even if the stresses were constant, the rate of shear would tend to zero as time increases; and hence even if convection was started it would not continue to bring in enough heat to replace the loss by conduction.

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1958 *February.*

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On the Equivalence of Saddle Point Approximations and Ray Theory in Elastic Wave Problems

S. D. Chopra

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Summary

It has been known for some time that the Bromwich method of expansion in negative powers of exponentials of the formal solutions of elastic wave problems for stratified media by wave theory methods gives rise to the various reflected and refracted waves predicted by the ray theory of geometrical optics. It is shown here by application to an example how the saddle point evaluation by the steepest descents method of the various terms in such an expansion of the solution leads to exactly the same expressions for the displacements in the reflected and refracted waves as can be obtained by the geometrical theory. The example chosen is that of the elastic waves produced by a harmonic point source of dilatational waves situated in the middle of an internal stratum bounded on both sides by halfspaces of identical elastic properties, all perfectly elastic, homogeneous and isotropic.

In the course of the demonstration, we give the reflection and refraction coefficients for the amplitudes of displacements when a plane P or SV wave is incident at an interface between two elastic media. These are in the form in which they occur in the wave solutions of problems involving reflection and refraction at plane interfaces.

1. Introduction

Bromwich (1916, p. 425–428) studied the problem of a bar with one end fixed and struck at the other by a particle moving in the direction of the length of the bar. He showed that the expansion of the formal operational solution in negative powers of exponentials leads to a solution in terms of pulses. The method has since been applied by Muskat (1938a, b), Pekeris (1948, p. 46 *et seq.*), Officer (1951), Newlands (1952, p. 222 *et seq.*), Honda & Nakamura (1953, 1954) and others to the formal solutions of elastic wave problems for stratified media.

Whereas Muskat and others studied problems involving plane boundaries, van der Pol & Bremmer (1937) have discussed the propagation of radio waves from a spherical point source situated above a spherical earth. They show (1937, pp. 844–6; see Bremmer, 1949, Ch. V for further detail) that the saddle point approximation for the reflected field gives an expression which may be obtained from the primary field by multiplying it by a reflection coefficient corresponding to the extra divergence acquired by a thin cone of rays on being reflected from a spherical instead of a plane surface. At the same time, the phase factor has to be

modified to take account of the distance travelled by the wave. A similar equivalence of the saddle point approximation with the ray theory has been shown also by Gerjuoy (1948, 1953) for a medium consisting of two fluid halfspaces with a plane interface and a point source of compressional waves in one of the halfspaces.

We propose to apply the method used by van der Pol & Bremmer to a problem involving reflection and refraction at plane interfaces and show how the saddle point evaluation of the various terms in the Bromwich expansion of the wave solution is equivalent to the geometric ray-path solution. We give here the dilatational part of the displacement potential ϕ_1, in one of the halfspaces, of the disturbance due to a harmonic point source of dilatational waves situated in the middle of an internal stratum bounded on both sides by two identical halfspaces with which it is in welded contact. As a preliminary to the Bromwich expansion of ϕ_1 in a form which brings out the intimate relationship between the wave solution and the geometric theory, we give the reflection and refraction coefficients when a plane wave is incident at an interface between two elastic media. We next give the expansion of ϕ_1 in terms of these coefficients. Finally, evaluating the displacements corresponding to two of the terms of this expansion at the saddle point by the method of steepest descents and also obtaining the expressions for the corresponding rays by the geometrical theory, we show that the two methods lead to identical expressions for each term. In the geometrical derivation of these expressions we must take account of the fact that when rays are transformed by reflection at an interface from the dilatational to the distortional type or *vice versa* or are refracted from one medium into the other, they no longer diverge from a point but from a circle round the axis of symmetry.

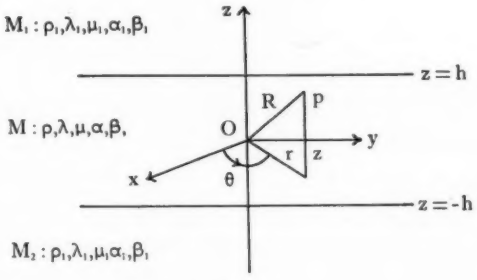

FIG. 1.

2. Formal solution for a compressional point source in an internal stratum

We assume the stratum M and the halfspaces M_1 and M_2 to be homogeneous, isotropic and perfectly elastic. Let

ρ = density, λ, μ = Lamé's constants,
$\alpha = \{(\lambda+2\mu)/\rho\}^{\frac{1}{2}}$ = velocity of dilatational waves,
$\beta = (\mu/\rho)^{\frac{1}{2}}$ = velocity of distortional waves,

for the stratum. The corresponding quantities for the halfspaces M_1 and M_2 (of identical elastic properties) are denoted by the same symbols with the suffix 1. Let $2h$ be the thickness of the stratum.

Take the origin at the source O which lies midway between the two interfaces and the axis of z perpendicular to the two interfaces with its positive direction

pointing into M_1. On account of axial symmetry about Oz, we use cylindrical polar coordinates r, θ, z. All quantities are independent of θ so that $\partial/\partial\theta \equiv 0$. Let U, V, W be the displacement components in the r, θ, z directions. For a point source of dilatational waves, $V = 0$. We introduce displacement potentials ϕ, ψ so that

$$U = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}, \qquad W = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r}; \tag{1}$$

ϕ, ψ determine the dilatational and distortional parts respectively of the disturbance and satisfy the equations

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}, \tag{2}$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}, \tag{3}$$

in M. If ϕ_1, ψ_1; ϕ_2, ψ_2 denote the corresponding potentials for M_1, M_2, then these satisfy similar equations with α, β replaced by α_1, β_1 respectively.

For a harmonic source, the time factor is $e^{i\omega t}$. This factor will be omitted in future from all expressions for potentials and displacements.

If $R = (r^2 + z^2)^{\frac{1}{2}}$, then the potential ϕ_0 for a point source of compressional waves at O can be represented by Sommerfeld's integral (Magnus & Oberhettinger, 1954, p. 34)

$$\phi_0 = \frac{1}{R}e^{-i\omega R/\alpha} = \int_0^\infty J_0(r\zeta)e^{-|z|a}\frac{\zeta\,d\zeta}{a}, \tag{4}$$

where $a = (\zeta^2 - \omega^2/\alpha^2)^{\frac{1}{2}}$ and J_0 is the Bessel function of the first kind and zero order. The representation (4) is valid for

$$-\tfrac{1}{2}\pi < \arg a \leqslant \tfrac{1}{2}\pi \quad \text{when} \quad -\pi < \arg(\omega/\alpha) \leqslant 0. \tag{5}$$

To satisfy the boundary conditions we combine with (4) suitable elementary solutions of the differential equations (2), (3) and their analogues for the halfspaces M_1, M_2. These solutions, obtainable by the method of separation of variables, are so chosen as to take account of the symmetry about the plane $z = 0$. Then the boundary conditions, viz. the continuity of displacements and stresses across the interfaces $z = \pm h$, lead to four distinct equations. Solving these equations we get the values of the unknowns involved in ϕ, ψ etc. Omitting the details of these calculations, we obtain

$$\phi_1 = \int_0^\infty \frac{(N_1 + N_1' e^{-2hb})\,e^{-ha-(z-h)a_1}}{Y_1 - Y_2 e^{-2ha} - Y_3 e^{-2hb} - Y_4 e^{-2ha-2hb}} J_0(r\zeta)\frac{\zeta\,d\zeta}{a}, \tag{6}$$

where we use the following notation:

$$\kappa_\alpha = \omega/\alpha, \quad \kappa_\beta = \omega/\beta, \quad \kappa_{\alpha_1} = \omega/\alpha_1, \quad \kappa_{\beta_1} = \omega/\beta_1, \tag{7}$$

$$a = (\zeta^2 - \kappa_\alpha^2)^{\frac{1}{2}}, \quad b = (\zeta^2 - \kappa_\beta^2)^{\frac{1}{2}}, \quad a_1 = (\zeta^2 - \kappa_{\alpha_1}^2)^{\frac{1}{2}}, \quad b_1 = (\zeta^2 - \kappa_{\beta_1}^2)^{\frac{1}{2}}, \tag{8}$$

$$\Omega = \zeta^2 - \tfrac{1}{2}\kappa_\beta^2, \quad \Omega_1 = \zeta^2 - \tfrac{1}{2}\kappa_{\beta_1}^2, \tag{9}$$

$$N_1 = \mu\kappa_\beta^2 a[(\mu_1\Omega_1 - \mu\zeta^2)b + (\mu\Omega - \mu_1\zeta^2)b_1], \tag{10}$$

$$N_1' = \mu\kappa_\beta^2 a[(\mu_1\Omega_1 - \mu\zeta^2)b - (\mu\Omega - \mu_1\zeta^2)b_1], \tag{11}$$

$$\begin{aligned} Y_1 = {} & \mu^2(\Omega^2 - \zeta^2 ab)(\zeta^2 - a_1b_1) - \tfrac{1}{4}\mu\mu_1\kappa_\beta^2\kappa_{\beta_1}^2(ab_1 + a_1b) - \\ & - 2\mu\mu_1\zeta^2(\Omega - ab)(\Omega_1 - a_1b_1) + \mu_1^2(\zeta^2 - ab)(\Omega_1^2 - \zeta^2 a_1b_1), \end{aligned} \tag{12}$$

$$\begin{aligned} Y_2 = {} & \mu^2(-\Omega^2 - \zeta^2 ab)(\zeta^2 - a_1b_1) - \tfrac{1}{4}\mu\mu_1\kappa_\beta^2\kappa_{\beta_1}^2(ab_1 - a_1b) - \\ & - 2\mu\mu_1\zeta^2(-\Omega - ab)(\Omega_1 - a_1b_1) + \mu_1^2(-\zeta^2 - ab)(\Omega_1^2 - \zeta^2 a_1b_1) \end{aligned} \tag{13}$$

$$\begin{aligned} Y_3 = {} & \mu^2(\Omega^2 + \zeta^2 ab)(\zeta^2 - a_1b_1) - \tfrac{1}{4}\mu\mu_1\kappa_\beta^2\kappa_{\beta_1}^2(ab_1 - a_1b) - \\ & - 2\mu\mu_1\zeta^2(\Omega + ab)(\Omega_1 - a_1b_1) + \mu_1^2(\zeta^2 + ab)(\Omega_1^2 - \zeta^2 a_1b_1) \end{aligned} \tag{14}$$

$$\begin{aligned} Y_4 = {} & \mu^2(\Omega^2 - \zeta^2 ab)(\zeta^2 - a_1b_1) - \tfrac{1}{4}\mu\mu_1\kappa_\beta^2\kappa_{\beta_1}{}^2(-ab_1 - a_1b) - \\ & - 2\mu\mu_1\zeta^2(\Omega - ab)(\Omega_1 - a_1b_1) + \mu_1^2(\zeta^2 - ab)(\Omega_1^2 - \zeta^2 a_1b_1). \end{aligned} \tag{15}$$

The radicals b, a_1, b_1 are subject to conditions similar to (5). We shall also use the additional notation:

$$N_2 = \mu\kappa_\beta^2\zeta a[(\mu - \mu_1)a_1b + (\mu_1\Omega_1 - \mu\Omega)] \tag{16}$$

$$N_3 = \mu\kappa_\beta^2\zeta b[(\mu\Omega - \mu_1\Omega_1) + (\mu_1 - \mu)ab_1] \tag{17}$$

$$N_4 = \mu\kappa_\beta^2 b[(\mu\Omega - \mu_1\zeta^2)a_1 + (\mu_1\Omega_1 - \mu\zeta^2)a] \tag{18}$$

$$N_5 = 2a\zeta[\mu^2\Omega(\zeta^2 - a_1b_1) - \mu\mu_1(\Omega + \zeta^2)(\Omega_1 - a_1b_1) + \mu_1^2(\Omega_1^2 - \zeta^2 a_1b_1)] \tag{19}$$

$$N_6 = 2b\zeta[\mu^2\Omega(\zeta^2 - a_1b_1) - \mu\mu_1(\Omega + \zeta^2)(\Omega_1 - a_1b_1) + \mu_1^2(\Omega_1^2 - \zeta^2 a_1b_1)]. \tag{20}$$

The following relations between the Y's and N's are easily verified.

$$\left.\begin{aligned} bN_5 &= aN_6 \\ N_1Y_3 + N_1'Y_1 &= -N_3N_5 \\ Y_1Y_4 + Y_2Y_3 &= -N_5N_6. \end{aligned}\right\} \tag{21}$$

Before proceeding with the expansion of ϕ_1, we give in the next paragraph the reflection and refraction coefficients for plane harmonic waves incident at a plane boundary between two elastic media and show how the expressions Y_1, Y_2, Y_3, N_1, N_2, N_3, N_4, N_5, N_6 arise there. The expressions Y_4 and N_1' do not arise directly in these coefficients but are connected with the others by means of the relations (21).

3. Reflection and refraction coefficients for plane waves

The general equations for reflection and refraction of plane waves at a plane interface between two elastic solids were given by Knott (1899, p. 92–96) and Zoeppritz (1919). They did not, however, solve these equations except in a few simple cases. Since then many writers have given the results of detailed numerical calculations in special cases. Blut (1932), following the method of Zoeppritz,

and Muskat & Meres (1940) give the reflection and refraction coefficients for plane waves incident at an interface between two elastic media and also give results of extensive numerical calculations. The form of their results, however, is not suited to our present purpose and it is therefore proposed to obtain these coefficients afresh, omitting the details.

Let H, H_1 be the two media with their elastic constants denoted by the same symbols as for the media M, M_1 of Section 2. In applying the results of the present section, which hold for any pair of media, we identify H, H_1 with the media M, M_1 respectively. Therefore, here and later, we employ the same symbols as in Section 2 to be able to use without change the notation defined in (7)–(20).

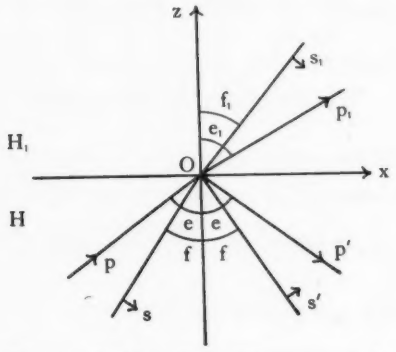

FIG. 2.—Incident P or SV wave and the reflected and refracted waves.

We take the interface between H, H_1 as the xOy plane and the z-axis drawn into H_1. Let motion take place in the xOz plane, and the waves propagate parallel to this plane. Then the displacement in the y-direction is zero and all quantities are independent of y, so that $\partial/\partial y \equiv 0$. If U, W be the displacement components along Ox, Oz respectively, then we can introduce displacement potentials Φ, Ψ, such that

$$U = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial z}, \qquad W = \frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x}; \tag{22}$$

Φ, Ψ give the dilatational and the distortional parts respectively of the motion.

An incident dilatational (P) or distortional (S) wave in H gives rise, in general, to four waves, a reflected dilatational (P') and a reflected distortional (S') in H, and a refracted dilatational (P_1) and a refracted distortional (S_1) in H_1. The S waves considered here are of the SV type that is, they are polarized in a plane perpendicular to the interfaces, but for brevity in writing we omit the V from our notation.

Let e and f denote the angles of incidence and reflection of the P and S waves respectively in H, and e_1, f_1, the angles of refraction into H_1. Then the boundary conditions—continuity of displacements and stresses across the interface $z = 0$—imply Snell's law

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta} = \frac{\sin e_1}{\alpha_1} = \frac{\sin f_1}{\beta_1} = \frac{\zeta}{\omega} \text{ (say)}. \tag{23}$$

ω is the circular frequency of the plane harmonic waves under consideration and ζ is defined by (23). We shall identify this ζ later with the ζ of Section 2. With ζ given by (23), we also have

$$a = i\kappa_\alpha \cos e, \quad b = i\kappa_\beta \cos f, \quad a_1 = i\kappa_{\alpha_1} \cos e_1, \quad b_1 = i\kappa_{\beta_1} \cos f_1, \tag{24}$$

where the symbols a, κ_α, etc. are defined in (7) and (8). Under certain conditions, some of the angles e, f, e_1, f_1 become complex. Later we will also allow ζ to become complex. In all these cases, the values of the radicals a, b, a_1, b_1 remain subject to the condition

$$-\tfrac{1}{2}\pi < \arg a, b, a_1, b_1 \leqslant \tfrac{1}{2}\pi. \tag{25}$$

We can now state our results in terms of the notation of (7)–(20).

Let $R_{PP'}$ denote the reflection coefficient for the complex amplitudes of displacements when P′ is derived from an incident P by reflection. Other reflection and refraction coefficients are denoted by similar symbols. Following Byerly (1942, p. 164), we adopt the convention that the directions of the displacements in the various waves are taken in such a way that all have positive components along Ox. These directions are indicated by arrows in Figure 2.

Incident P wave

For a P wave incident in the third quadrant in H, suitable values of the potentials are

$$\left.\begin{aligned} \Phi &= A \exp\{-i\kappa_\alpha(x \sin e + z \cos e)\} + A' \exp\{-i\kappa_\alpha(x \sin e - z \cos e)\} \\ \Psi &= B' \exp\{-i\kappa_\beta(x \sin f - z \cos f)\} \end{aligned}\right\} \text{ in } H, \tag{26}$$

$$\left.\begin{aligned} \Phi_1 &= A_1 \exp\{-i\kappa_{\alpha_1}(x \sin e_1 + z \cos e_1)\} \\ \Psi_1 &= B_1 \exp\{-i\kappa_{\beta_1}(x \sin f_1 + z \cos f_1)\} \end{aligned}\right\} \text{ in } H_1. \tag{27}$$

The time factor $e^{i\omega t}$ is omitted in all these.

Applying the boundary conditions and solving the resulting equations, we obtain for the reflection and refraction coefficients

$$\left.\begin{aligned} R_{PP'} &= \frac{Y_2}{Y_1}, & R_{PP_1} &= \frac{\alpha}{\alpha_1}\frac{N_1}{Y_1} \\ R_{PS'} &= \frac{i\alpha}{\beta}\frac{N_5}{Y_1}, & R_{PS_1} &= \frac{i\alpha}{\beta_1}\frac{N_2}{Y_1} \end{aligned}\right\} \tag{28}$$

Incident SV wave

For an SV incident in H, suitable values for the potentials are

$$\left.\begin{aligned} \Phi &= A' \exp\{-i\kappa_\alpha(x \sin e - z \cos e)\}, \\ \Psi &= B \exp\{-i\kappa_\beta(x \sin f + z \cos f)\} + B' \exp\{-i\kappa_\beta(x \sin f - z \cos f)\}, \end{aligned}\right\} \text{ in } H, \tag{29}$$

$$\left.\begin{aligned} \Phi_1 &= A_1 \exp\{-i\kappa_{\alpha_1}(x \sin e_1 + z \cos e_1)\}, \\ \Psi_1 &= B_1 \exp\{-i\kappa_{\beta_1}(x \sin f_1 + z \cos f_1)\} \end{aligned}\right\} \text{ in } H_1, \tag{30}$$

and the resulting values of the reflection and refraction coefficients are

$$\left.\begin{aligned} R_{SP'} &= \frac{i\beta N_6}{\alpha Y_1}, & R_{SP_1} &= \frac{i\beta N_3}{\alpha_1 Y_1} \\ R_{SS'} &= \frac{Y_3}{Y_1}, & R_{SS_1} &= \frac{\beta N_4}{\beta_1 Y_1}. \end{aligned}\right\} \tag{31}$$

4. Identification of the two definitions of ζ

We now show how the ζ of (23) may be identified with the ζ of Section 2 and thus prepare for the use of the above reflection and refraction coefficients in the Bromwich expansion of ϕ_1.

Weyl (1919) has shown that, in a halfspace like $z \geqslant 0$, Sommerfeld's integral (4) for the spherical point source may be regarded as the superposition of all plane waves of equal intensity whose directions of propagation make with the z-axis not only all angles e lying between 0 and $\frac{1}{2}\pi$ but also a continued series of complex angles extending from $\frac{1}{2}\pi$ to $\frac{1}{2}\pi + i\infty$. If g be the second polar angle of the direction of propagation of any such wave, then the superposition of all these waves gives the expression

$$\int_{0,\frac{1}{2}\pi}^{\frac{1}{2}\pi+i\infty} \int_{0}^{2\pi} \exp\{-i\kappa_\alpha(x\sin e\cos g + y\sin e\sin g + z\cos e)\}\, E\sin e\, de\, dg, \tag{32}$$

where E is a constant determining the common intensity of the waves and $\sin e\, de\, dg$ is the element of solid angle. Carrying out the integration with respect to g gives

$$2\pi E \int_{0,\frac{1}{2}\pi}^{\frac{1}{2}\pi+i\infty} J_0(r\kappa_\alpha \sin e)\exp(-i\kappa_\alpha z\cos e)\sin e\, de. \tag{33}$$

Putting $\kappa_\alpha \sin e = \zeta$ and taking $E = \kappa_\alpha/2\pi i$ leads to Sommerfeld's integral. As the substitution $\kappa_\alpha \sin e = \zeta$ is the same as (23), and both here and in (23), e is the angle made by the corresponding plane wave with the z-axis, the ζ of (23) is identified with that of Section 2.

This identification indicates that we shall also need to consider complex values for the angles of incidence, reflection and refraction. In the next paragraph, we allow ζ to assume general complex values. Then e, f, e_1, f_1 will also have general complex values, without a simple physical picture. We then define the coefficients $R_{PP'}$ etc. by means of the relations (28) and (31) subject to the condition (25) which renders them single-valued functions of ζ.

5. Bromwich expansion of ϕ_1

The integral in ϕ_1 is of the type

$$\int_0^\infty \zeta F(\zeta) J_0(r\zeta)\, d\zeta, \tag{34}$$

where $F(\zeta)$ is an even function of ζ. We now regard ζ as a complex variable and $\mathrm{Re}(\omega) > 0$. We already have $\mathrm{Im}(\omega) \leqslant 0$ by (5). Making use of the relation

$$J_0(r\zeta) = \tfrac{1}{2}Hs_0(r\zeta) + \tfrac{1}{2}Hi_0(r\zeta), \tag{35}$$

where H_{s0} and H_{i0} are Hankel functions of the first and second kinds respectively and of zero order (Jeffreys & Jeffreys 1956, Sec. 21.02), we can transform the path of integration from the real axis into the Sommerfeld contour Γ (Figure 3)*, so that

$$\phi_1 = \int_0^\infty \zeta F(\zeta) J_0(r\zeta)\, d\zeta = \tfrac{1}{2} \int_\Gamma \zeta F(\zeta)\, Hi_0(r\zeta)\, d\zeta. \tag{36}$$

Γ starts at $-i\infty$ in the 4th quadrant and after surrounding the branch lines $\mathrm{Re}(a) = 0$, $\mathrm{Re}(b) = 0$, $\mathrm{Re}(a_1) = 0$, $\mathrm{Re}(b_1) = 0$ and all poles of $F(\zeta)$ to the right of

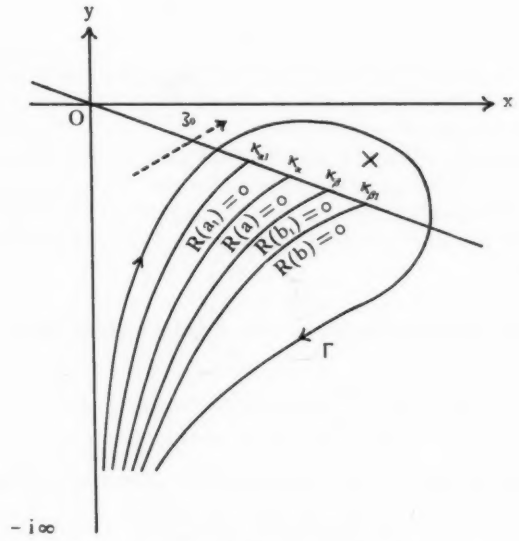

FIG. 3.—Sommerfeld contour Γ in the ζ-plane.

the imaginary axis, returns to $-i\infty$. A typical such pole is shown by a cross in Figure 3. For details of the transformation, see Lapwood (1949, p. 73) or Newlands (1952, p. 222 *et seq.*). We thus have

$$\phi_1 = \tfrac{1}{2} \int_\Gamma \frac{(N_1 + N_1' e^{-2\hbar b}) e^{-\hbar a - (z-h)a_1}}{Y_1 - Y_2 e^{-2\hbar a} - Y_3 e^{-2\hbar b} - Y_4 e^{-2\hbar a - 2\hbar b}} Hi_0(r\zeta) \frac{\zeta\, d\zeta}{a}. \tag{37}$$

If we take $|\omega|$ large enough, then $|(Y_2/Y_1)e^{-2\hbar a}|$, $|(Y_3/Y_1)e^{-2\hbar b}|$ and $|(Y_4/Y_1)e^{-2\hbar a - 2\hbar b}|$ are all < 1 and we can expand the integrand in ascending powers of these exponentials. This expansion is uniformly convergent and we can integrate it term by term. We thus get

* This and other figures are drawn for the case $\alpha_1 > \alpha > \beta_1 > \beta$, but the argument in the text is valid for all cases.

$$\phi_1 = \tfrac{1}{2}\int_\Gamma \Big\{ \frac{N_1}{Y_1} + \frac{N_1 Y_2}{Y_1^2} e^{-2ha} + \frac{N_1 Y_3 + N_1' Y_1}{Y_1^2} e^{-2hb} + \frac{N_1 Y_2^2}{Y_1^3} e^{-4ha} + \\ + \frac{(N_1 Y_3 + N_1' Y_1) Y_3}{Y_1^3} e^{-4hb} + \frac{(N_1 Y_3 + N_1' Y_1) Y_2 + N_1 (Y_1 Y_4 + Y_2 Y_3)}{Y_1^3} \times \\ \times e^{-2ha-2hb} + \ldots \Big\} e^{-ha-(z-h)a_1} H i_0(r\zeta) \frac{\zeta\, d\zeta}{a}$$

$$= \tfrac{1}{2}\int_\Gamma \{ R_{PP_1} + R_{PP} R_{PP_1} e^{-2ha} + R_{PS} R_{SP_1} e^{-2hb} + R_{PP_1}^2 R_{PP_1} e^{-4ha} + \\ + R_{PS} R_{SS} R_{SP_1} e^{-4hb} + (R_{PP} R_{PS} R_{SP_1} + R_{PS} R_{SP} R_{PP_1}) e^{-2ha-2hb} + \\ + \ldots \} e^{-ha-(z-h)a_1} H i_0(r\zeta) \frac{\alpha_1 \zeta\, d\zeta}{\alpha a}, \qquad (38)$$

making use of the relations (21) and the reflection and refraction coefficients (28), (31). This is the required Bromwich expansion of ϕ_1. We shall denote the successive terms in the expansion by ϕ_{11}, ϕ_{12}, etc.

We now show in Sections 6–8 that the evaluation of these terms by the saddle point method leads to the same expressions for the displacements as can be obtained for the corresponding rays by the fully developed geometrical theory. We shall illustrate by evaluating by both methods the displacements corresponding to ϕ_{11} and ϕ_{13} only.

6. Saddle point approximation to the displacement given by ϕ_{11}

From (38),

$$\phi_{11} = \tfrac{1}{2}\int_\Gamma \frac{\alpha_1 \zeta}{\alpha a} R_{PP_1} e^{-ha-(z-h)a_1} H_{i0}(r\zeta)\, d\zeta. \qquad (39)$$

We first replace $H_{i0}(r\zeta)$ by the first term of its asymptotic expansion, viz.

$$H_{i0}(r\zeta) \sim \left(\frac{2}{\pi r\zeta}\right)^{\frac{1}{2}} e^{-i(r\zeta - \frac{1}{4}\pi)} \qquad (40)$$

valid for $-2\pi < \arg(r\zeta) < \pi$ and $|r\zeta|$ large (Copson, 1935, p. 336). We then get

$$\phi_{11} = \frac{\alpha_1}{\alpha}\left(\frac{i}{2\pi r}\right)^{\frac{1}{2}} \int_\Gamma \frac{\zeta^{\frac{1}{2}}}{a} R_{PP_1} e^{-ir\zeta - ha - (z-h)a_1}\, d\zeta. \qquad (41)$$

The component W_{11} along Oz of the displacement given by ϕ_{11} is

$$W_{11} = \frac{\partial \phi_{11}}{\partial z} \sim -\frac{\alpha_1}{\alpha}\left(\frac{i}{2\pi r}\right)^{\frac{1}{2}} \int_\Gamma \frac{a_1 \zeta^{\frac{1}{2}}}{a} R_{PP_1} e^{-ir\zeta - ha - (z-h)a_1}\, d\zeta. \qquad (42)$$

An account of the method of steepest descents is given by Jeffreys & Jeffreys (1956, Section 17.04). For the evaluation of the integral (42), we shall use the formula given by them, viz., for large positive r,

$$\int_A^B \mathrm{X}(\zeta)\, e^{rf(\zeta)}\, d\zeta \sim \frac{\mathrm{X}(\zeta_0)\, e^{rf(\zeta_0)} (2\pi)^{\frac{1}{2}} e^{i\epsilon}}{|rf''(\zeta_0)|^{\frac{1}{2}}}, \qquad (43)$$

where ζ_0 is the saddle point and ϵ is the angle made with the real axis by the path of steepest descents through ζ_0. For its applicability to (42), we must have $r \gg z$.

We write

$$f(\zeta) = -ir\zeta - ha - (z-h)a_1, \tag{44}$$

then the saddle point is given by

$$\frac{\partial f}{\partial \zeta} = -ir - \frac{h\zeta}{(\zeta^2-\kappa_\alpha{}^2)^{\frac{1}{2}}} - \frac{(z-h)\zeta}{(\zeta^2-\kappa_{\alpha_1}{}^2)^{\frac{1}{2}}} = 0. \tag{45}$$

Let ζ_0 be the root of (45) and e_0, e_{10} the values of e, e_1 corresponding to $\zeta = \zeta_0$ as given by (23), then

$$\frac{\zeta_0}{\omega} = \frac{\sin e_0}{\alpha} = \frac{\sin e_{10}}{\alpha_1}, \tag{46}$$

and (45) becomes

$$h \tan e_0 + (z-h) \tan e_{10} = r. \tag{47}$$

(46) and (47) give real values for the angles e_0 and e_{10} which correspond to the minimum-time-path ray PP_1 of geometrical optics. This ray starts at the source

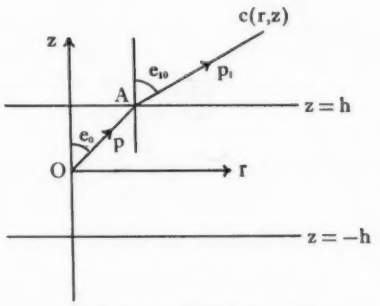

FIG. 4.—The Ray PP_1.

O (Figure 4) as a P wave at an angle e_0 with Oz and after being refracted at A at the interface $z = h$ arrives at the point $C(r, z)$ as a P_1 wave at an angle e_{10} with Oz. Since $r \gg z$, e_0 or e_{10} is close to 90° according as $\alpha > \alpha_1$ or $\alpha < \alpha_1$.

The path of steepest descents through ζ_0 is given by the condition that

$$\tfrac{1}{2}(\zeta-\zeta_0)^2\left(\frac{\partial^2 f}{\partial \zeta^2}\right)_{\zeta=\zeta_0} \text{ is real and negative.} \tag{48}$$

We have

$$\left(\frac{\partial^2 f}{\partial \zeta^2}\right)_{\zeta=\zeta_0} = \frac{i}{\omega}\{h\alpha \sec^3 e_0 + (z-h)\alpha_1 \sec^3 e_{10}\}. \tag{49}$$

Let $\arg \omega = \arg \kappa_{\alpha_1} = \arg \kappa_\alpha = -\delta$, so that $0 \leqslant \delta < \frac{1}{2}\pi$. Then, by condition (48), the path of steepest descent through ζ_0 makes an angle $\frac{1}{4}\pi - \frac{1}{2}\delta$ with the real axis. The direction of the path of steepest descent through ζ_0 is shown by a dotted line in Figure 3. The modification of the path Γ to pass through ζ_0 in this direction is evidently possible.

The saddle point approximation to (42) therefore gives

$$W_{11} \sim -\frac{i\omega \sin e_0 \cos e_{10}(R_{PP_1})_{\zeta=\zeta_s} \exp\left[-i\omega\left\{\frac{h \sec e_0}{\alpha}+\frac{(z-h)\sec e_{10}}{\alpha_1}\right\}\right]}{\alpha \cos e_0[r\{h \tan e_0 \sec^2 e_0+(z-h)\tan e_{10}\sec^2 e_{10}\}]^{\frac{1}{2}}}. \quad (50)$$

Since this is the component along Oz of the displacement in a dilatational wave which is travelling at C in a direction making an angle e_{10} with Oz, the displacement in the wave is obtained by simply omitting the factor $\cos e_{10}$ in (50).

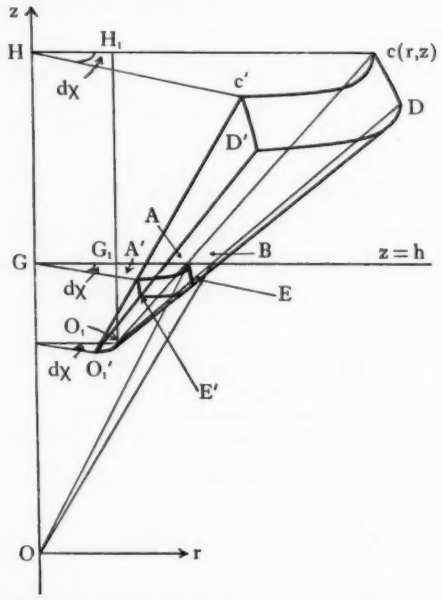

FIG. 5.—Geometrical derivation of the displacement in the ray PP_1.
$LAOG = e$, $LBOG = e+de$, $LAO_1G_1 = e_1$, $LBO_1G_1 = e_1+de_1$.
$OH = z$, $HC = r = h \tan e+(z-h)\tan e_1$, $O_1G_1 = h_1$, $O_1H_1 = h_1+z-h$.
CC', DD', AA', EE' are arcs of circles about Oz; CD, AE about O_1; $C'D'$, $A'E'$ about O_1'.

7. Geometrical derivation of the displacement in the ray PP_1

Let a ray starting at the source O (Figure 5) as a P wave at an angle e to Oz meet the interface $z = h$ at A, be refracted into M_1 at an angle e_1 according to Snell's law and be observed at the point $C(r, z)$. Let a neighbouring ray in the plane zOA leave the source at an angle $e+de$ to Oz, meet the interface at B, and be refracted into M_1 at the angle e_1+de_1. The rays arriving at C and the neighbouring point D appear to diverge not from O but from O_1 where the refracted rays AC and BD meet. Let $O_1G_1 = h_1$ be the perpendicular from O_1 to $z = h$. We need the value of h_1. The equation of AC is

$$z-h = (r-h\tan e)\cot e_1. \quad (51)$$

The position of O_1 is obtained by solving (51) with the equation

$$0 = -h\sec^2 e \cot e_1-(r-h\tan e)\operatorname{cosec}^2 e_1 \cdot \frac{de_1}{de} \quad (52)$$

obtained by differentiating (51) with respect to e. We have also

$$\frac{de_1}{de} = \frac{\sin e_1 \cos e}{\cos e_1 \sin e} \tag{53}$$

from Snell's law $\alpha \sin e_1 = \alpha_1 \sin e$. These give

$$h_1 = \frac{\sin e \cos^3 e_1}{\cos^3 e \sin e_1} h. \tag{54}$$

The P wave starting at O has been derived from the potential

$$\phi_0 = \frac{1}{R} \exp(-i\kappa_\alpha R), \tag{55}$$

where R is the distance from O. Hence the displacement in this wave at A in the stratum is given by

$$\frac{\partial \phi_0}{\partial R} \sim -\frac{i\omega}{\alpha R} \exp(-i\kappa_\alpha R) = -\frac{i\omega}{\alpha h \sec e} \exp(-i\kappa_\alpha R), \tag{56}$$

neglecting the term in $1/R^2$. This implies that $|\omega| \gg \alpha/R$, or, in other words, the distance OA is very large compared to the wave length of the waves under consideration. We can therefore regard the wave at A as a plane wave and use the reflection and refraction coefficients for plane waves given in (28), (31). The amplitude of the displacement given by (56) is

$$-(i\omega/\alpha h) \cos e. \tag{57}$$

Since the ray is refracted into M_1 as P_1, we have to multiply this by R_{PP_1}, so that the amplitude in the ray PP_1 at A in M_1 is

$$-(i\omega/\alpha h) \cos e \,.\, R_{PP_1}. \tag{58}$$

On account of axial symmetry about Oz, energy in the medium M_1 carried along by the ray AC and its neighbouring rays in the PP_1 wave diverges not from the point O_1 in a conical element, but in an element of volume formed by revolving the area O_1CD through a small angle $d\mathrm{X}$ about Oz. Hence the amplitudes in the ray PP_1 at A and C are in the limiting ratio of the inverse square roots of the areas $AEE'A'$ and $CDD'C'$, since the same amount of energy crosses these areas in unit time. We have

$$\text{area} \quad AEE'A' \doteqdot O_1A \, de_1 \, GA \, d\mathrm{X} = h_1 \sec e_1 \, h \tan e \, de_1 \, d\mathrm{X}, \tag{59}$$

and

$$\text{area} \quad CDD'C' \doteqdot O_1C \, de_1 \, HC \, d\mathrm{X} = (h_1 + z - h) \sec e_1 \, r \, de_1 \, d\mathrm{X}. \tag{60}$$

Hence the amplitude of the displacement at C is approximately

$$= \left(Lt \frac{\text{area } AEE'A'}{\text{area } CDD'A'}\right)^{\frac{1}{2}} \times \text{amplitude of the displacement at } A \text{ in } M_1$$

$$= -\frac{i\omega \sin e}{\alpha \cos e} R_{PP_1} [r\{h \tan e \sec^2 e + (z-h) \tan e_1 \sec^2 e_1\}]^{-\frac{1}{2}}. \tag{61}$$

The appropriate phase factor for the displacement at C in the ray PP_1 is

$$\exp\left[-i\omega\left(\frac{OA}{\alpha}+\frac{AC}{\alpha_1}\right)\right]=\exp\left[-i\omega\left\{\frac{h\sec e}{\alpha}+\frac{(z-h)\sec e_1}{\alpha_1}\right\}\right]. \quad (62)$$

Incorporating this into (61) we get the displacement at C in the ray PP_1. When e, e_1 are replaced by e_0, e_{10} respectively, this gives the same expression as given by (50).

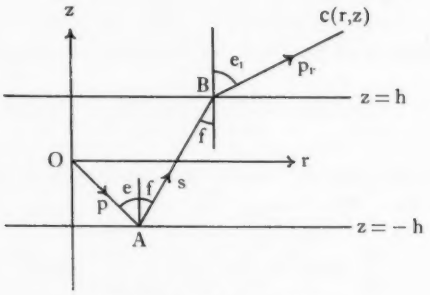

FIG. 6.—The Ray PSP_1.

8. Displacements in the ray PSP$_1$

It may be shown as in Section 6 that the displacement corresponding to ϕ_{13} given by the saddle point evaluation is

$$-\frac{i\omega\sin e R_{PS}R_{SP_1}\exp\left[-i\omega\left\{\frac{h\sec e}{\alpha}+\frac{2h\sec f}{\beta}+\frac{(z-h)\sec e_1}{\alpha_1}\right\}\right]}{\alpha\cos e[r\{h\tan e\sec^2 e+2h\tan f\sec^2 f+(z-h)\tan e_1\sec^2 e_1\}]^{\frac{1}{2}}}, \quad (63)$$

where the angles e, e_1, f are now given by the relations

$$\frac{\sin e}{\alpha}=\frac{\sin f}{\beta}=\frac{\sin e_1}{\alpha_1} \quad (64)$$

and

$$h\tan e+2h\tan f+(z-h)\tan e_1=r. \quad (65)$$

(64) and (65) give real values for the angles e, f and e_1 which correspond to the minimum-time-path ray PSP_1 of geometrical optics. The path of this ray is shown in Figure 6.

The amplitude of the displacement in the ray PSP_1 may be derived geometrically by a repeated application of the method of Section 7. In Figure 7, $OABC$ is the path of the ray PSP_1 and $ODEF$ that of a neighbouring ray in the plane zOA. The three-dimensional picture obtained by revolving the figure in the plane zOA through an angle $d\mathrm{X}$ about Oz is not shown in Figure 7 for the sake of clarity. The amplitude of the displacement at C is

$$=-\frac{i\omega}{\alpha h\sec e}R_{PS}\left(\frac{O'A.LA}{O'B.GB}\right)^{\frac{1}{2}}R_{SP_1}\left(\frac{O_1B.GB}{O_1C.HC}\right)^{\frac{1}{2}}$$

$$=-\frac{i\omega}{\alpha h\sec e}R_{PS}R_{SP_1}\left\{\frac{h'\sec f.h\tan e.h_1\sec e_1}{(h'+2h)\sec f(h_1+z-h)\sec e_1.r}\right\}^{\frac{1}{2}}. \quad (66)$$

But by an immediate extension of (54),

$$h' = \frac{\sin e \cos^3 f}{\sin f \cos^3 e} h, \qquad h_1 = \frac{\sin f \cos^3 e_1}{\sin e_1 \cos^3 f}(h' + 2h). \tag{67}$$

Substituting these values in (66), the amplitude of the displacement in the ray PSP_1 is

$$-\frac{i\omega \sin e \, R_{PS} R_{SP_1}}{\alpha \cos e [r\{h \tan e \sec^2 e + 2h \tan f \sec^2 f + (z-h)\tan e_1 \sec^2 e_1\}]^{\frac{1}{2}}}. \tag{68}$$

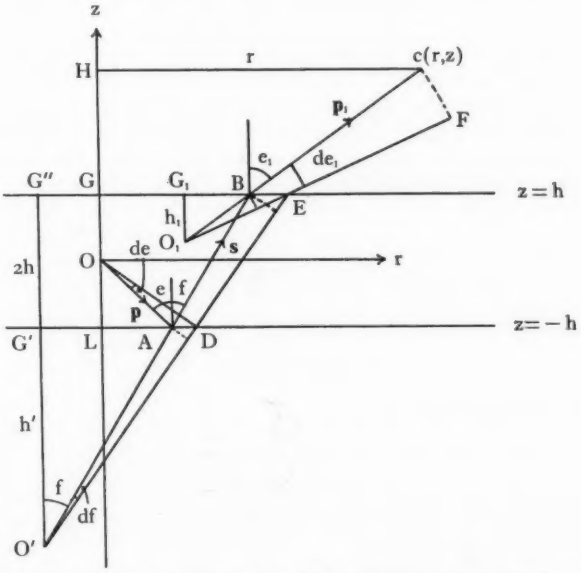

FIG. 7.—Geometrical derivation of the displacement in the ray PSP_1.
$OH = z$, $HC = r = h \tan e + 2h \tan f + (z-h) \tan e_1$.

The wave PSP_1 travels the distances OA, AB, BC with the velocities α, β, α_1 respectively and so the appropriate phase factor in the displacement in this wave at C is

$$\exp\left[-i\omega\left\{\frac{h \sec e}{\alpha} + \frac{2h \sec f}{\beta} + \frac{(z-h)\sec e_1}{\alpha_1}\right\}\right]. \tag{69}$$

Incorporating this factor into (68), we get the same expression as in (63).

9. Concluding remarks

The preceding work can be applied also to the other terms in the expansion of ϕ_1. We thus see that evaluating at the saddle point by the method of steepest descents gives the same results as obtained by the fully developed geometrical theory. Although we have illustrated the principle of the equivalence of the saddle point approximation with the geometrical theory by a single example, the method is applicable to all similar problems and the principle is true generally.

The Sommerfeld integral representation (4) for the field due to a point source is obtained by summing up plane waves of equal intensity and suitable directions.

The Bromwich expansions of the derived potentials ϕ_1, etc. exhibit these potentials as summations of plane waves that can be obtained by all possible reflections and refractions of the plane waves contained in (4). Of all the plane waves contained in any particular term of the expansion, the saddle point approximation of the term picks out that which gives the corresponding minimum-time-path ray of geometrical optics. We may recall here the remark made by Lapwood (1949, p. 80). Evaluating at the saddle point by the method of stationary phase, he gets a result which is identical with a result obtained earlier by Jeffreys (1926) for plane waves, and observes that this is "..., as it should be, since we have approximated at a particular value ω and a particular point ζ." We are here dealing with a fixed value of ω.

The importance of the saddle point approximation lies further in the fact that at the value of ζ picked out by it, the phase of the waves forming the integral has a stationary value. Here the neighbouring plane waves are nearly in phase and add to build up the amplitude. Away from this value of ζ, they fall more and more out of phase and there is no building up of the amplitude. The best example of this is the expressoin (4) for the spherical point source itself, viz.

$$\phi_0 = \int_0^\infty e^{-|z|a} J_0(r\zeta) \frac{\zeta \, d\zeta}{a} \sim \left(\frac{i}{2\pi r}\right)^{\frac{1}{2}} \int_\Gamma e^{-ir\zeta - |z|a} \frac{\zeta^{\frac{1}{2}} \, d\zeta}{a}. \tag{70}$$

for which the saddle point approximation gives the exact value

$$R^{-1} \exp(-i\kappa_\alpha R). \tag{71}$$

The saddle point is given by

$$\zeta = \kappa_\alpha \sin e, \tag{72}$$

where e is the angle made with Oz by the line joining the source to the point (r, z). Thus the value of ζ at the saddle point gives the direction of propagation from the source to the point (r, z).

The saddle point approximation to any term in the Bromwich expansion gives only a part of the disturbance resulting from that term. To get the complete disturbance, we must add the contributions arising from the poles and branch lines of the integrand that may have to be passed over in deforming the Sommerfeld contour to bring it into coincidence with the path of steepest descent. It is hoped to give a fuller account of the problem of the point source in an internal stratum at a later date.

10. Acknowledgments

The author is greatly indebted to Dr E. R. Lapwood for suggesting this investigation. The work presented here was done while the author was at the St John's College, Cambridge, as a Research Scholar under the Modified Overseas Scholarships Scheme of the Government of India as a sponsored scholar of the Panjab University. This work forms part of the dissertation submitted to the University of Cambridge for the Ph.D. Degree.

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Notes on Progress in Geophysics

Some Recent Developments in the Study of Microseisms in Great Britain and the United States

J. Darbyshire and H. M. Iyer

Summary

Although microseisms, the small earth tremors of period 4–10 s and amplitude up to 10μ, have been studied for nearly eighty years, many of the results and conclusions obtained are still controversial. The controversy has been centred mainly round the work done in Britain and the United States. An attempt will be made in this article to give an account of the present position of the work in both countries and to suggest ways and means by which agreement can be reached.

1. British work

This work has been largely inspired by the theory put forward by Longuet-Higgins (1950). This theory demonstrated how the pressure effect at the sea bottom due to a stationary sea wave system could be sufficient to generate microseisms. Such a system would be formed when trains of waves of the same period meet head on and this can happen near the coast when incident and reflected waves meet and also near the centre of a fast moving depression where the wind can veer round very quickly and trains of waves moving in opposite directions are generated. One important consequence of this theory is that the microseism period should be half the wave period as was originally observed by Bernard (1941). Subsequent British work (Deacon 1947, Darbyshire 1950, 1954) has confirmed this ratio but some American results do not fit so well.

The North Atlantic Ocean with its extensive meteorological coverage is an excellent testing ground for the use of microseisms. The results found can then be applied to other oceans like the Indian and the Pacific where there are far less weather data available. An attempt was thus made to track Atlantic depressions using microseisms. Some work on this had already been done by Lee in 1935 on the basis of microseisms recorded at Kew Observatory. He assumed that the microseisms were Rayleigh waves—waves where the ground particles move in ellipses contained by the vertical and the direction of propagation. In such a simple case, the tangent of the angle of approach of the waves would be given by the ratio of the amplitude of the E.W. and N.S. components. The right quadrant could be found from the phase relationships between the two horizontal and the vertical components. Lee did find some relation between the phase relationships and the quadrant of approach. Darbyshire (1954) tried to improve this method by assuming that microseisms consisted not only of Rayleigh waves but also of

Love waves where the ground particles move in a horizontal plane at right-angles to the direction of propagation. It would follow then that the vertical component would be free of the effect of Love waves and so records of the vertical and horizontal components were correlated while they were displaced in time over a definite range. From the point of time at which maximum correlation was obtained, the phase difference could be determined and from the actual value of the correlation coefficients the ratio of Rayleigh to Love wave activity and the angle of approach could be estimated. One storm in 1951 was tracked quite successfully by this method. Geddes (1957) also tracked a storm with two horizontal seismographs by considering the variations in the ratio of the amplitudes, assuming the co-existence of Love and Rayleigh waves.

More recent work has aimed mainly at improving the original correlation technique to suit all conditions of storm movement and position (Iyer 1958). A new refinement has been the use of band pass filters in analysing microseism records so that microseismic activity is divided into different period groups. This facilitates the study of microseisms caused by more than one source. A detailed study using this technique indicates the complexity of the nature of the microseismic source. Another interesting result is that the Love waves appear to have a random nature like background noise and come from many directions. The nature of Love waves and the variation of the Love/Rayleigh ratio with respect to the position of the generating source and recording station should hold interesting geological information about the nature of the Earth's crust.

All this work has not taken into account the effect of refraction of microseisms. It is still impossible to allow for this effect when the waves travel over land because of lack of knowledge of the elastic properties of the underlying crust but the effect while travelling across the sea can be estimated, at least in a qualitative manner, if it is assumed that they are waves between the ocean surface and the sea bed of the type first suggested by Stoneley (1926). Darbyshire (1956) worked out refraction diagrams for Bermuda on this basis and the diagrams explain some of the anomalies found by Carder (1955) for this island. Similar diagrams were prepared for the British Isles by Darbyshire & Darbyshire (1957) and Iyer & others (1958). This work has shown that the effect of refraction could explain in many cases the so-called microseism barriers because for certain directions, the sea depths form a configuration which acts like a diverging lens on the waves, the energy being spread over a wide angle.

The main development in the instrumental field in Britain has been the development by Tucker (1958) of a new horizontal seismograph which has been designed primarily for the study of microseisms. It incorporates an electronic feed-back circuit so that the instrumental characteristics are determined entirely by the constants in the electric circuit and are independent of the mechanical parts. The instrument has a flat response from 2 to 10 s period, while the magnification is 3000. The output can be fed into a pen-recorder. A detailed account of this instrument will be published shortly.

2. American work

The study of microseisms received a considerable impetus in the United States because they could be used to track the hurricanes which advance along the Atlantic coast from the Carribbean area. Considerable work on storm tracking was done by the Rev. B. Macelwane who used a tri-partite station with horizontal Macelwane-Sprengnether seismographs, with accurate timing devices for the three

stations. He was very successful in this connection and his methods were extended by the U.S. Navy so that between 1944 and 1947 many such tri-partite stations were in use. The tri-partite method is not completely satisfactory, however, as it is very difficult to identify individual waves on a seismograph trace when the distance between stations exceeds 10 miles (Shaw 1921), and for shorter distances, measurements of the order of a fraction of a second have to be made on waves of about 6 s period. A report by Donn & Blaik (1952) suggests that the success of the method has not been in proportion to the amount of money and effort spent.

The indifferent success of the tri-partite method led Marion Gilmore (1953) to develop an entirely empirical method. He argued that every storm giving rise to microseisms should produce a definite ratio of intensity at two stations depending on its own position. He claimed that this principle is independent of the nature, origin, composition and path of propagation of the waves. Data collected over eight years were used to prepare micro-ratio charts in which iso-ratio lines are drawn for pairs of stations.

The difficulties of this method consist in keeping seismograph characteristics constant over a number of years. Moreover, British work has shown that the intensity depends not only on the position but also on the strength of the storm and its history for some hours previously.

All the American work so far described has been done by using horizontal seismographs and these probably recorded a large proportion of Love wave activity which would have the characteristics of noise.

Future progress in storm tracking would appear to lie in combining the British correlation technique with the American tri-partite method. Research in England has hitherto been connected mainly with the three-component station working at Kew, but another three-component station has now been set up at Cambridge and correlation of the vertical and horizontal components of the two stations to ascertain the phase differences should give promising results.

The stationary wave theory has never been universally accepted in the U.S.A. Carder (1954) and Dinger & Fisher (1955) have produced evidence which supports it but other workers, notably, Donn (1954) do not accept the theory and from time to time produce evidence against it. This evidence is usually concerned with the two to one ratio and Donn concludes that the microseism period is a function of the depth of the sea at the storm position. It is, however, very difficult to draw any conclusions from observations which do not include both the sea wave and microseism spectra. The sea wave spectrum is usually quite wide and there can be a selective absorption of wave energy in a narrower frequency range to form microseisms. There are various ways in which this can be brought about. It follows from Longuet-Higgins's theory that certain frequencies would resonate with various depths of water rather like the organ pipe effect. There is also the effect of refraction so that waves of one frequency might be concentrated at a particular locality. It would be advisable to prepare a refraction diagram for the east coast of the United States similar to the one prepared for the British Isles. It is very likely that this would explain many of the results found by Donn. Some discrepancies from the two to one ratio may also occur due to the action of swell in shallow water areas where ordinary progressive sea waves have some effect on the sea bottom. This effect would, of course, be of the same period as the swell. This possibility was suggested by Bernard (1952).

American work on the Pacific side has been led by Gutenburg who is also very critical of the stationary wave theory. The same objections can be made to this

work as in the work of Donn. In the Pacific case, however, the situations are even more difficult to deal with as the microseism source is always very far away and direct microseisms from the source are very rarely observed. Interesting results have been obtained from the three-component station at Palomar which uses strain seismographs which can readily distinguish between Rayleigh and Love waves. The results confirm the existence of both type of wave in microseisms but the Love waves often come from the same direction as the Rayleigh waves and do not form a background noise as shown by recent British work. This discrepancy may be due to the fact that these microseisms are mainly caused by swell off the coast and have not traversed a long ocean path.

The only theory that can be regarded as a serious rival to the stationary wave theory is that of Press & Ewing (1948). This postulates that there are certain periods set up between the ocean surface and the sea bed which are associated with stationary values of group velocity. These periods could be excited by atmospheric oscillations of the same period. This system demands, however, a long homogeneous path which is very rarely obtained in practice. Although this theory would appear quite plausible in some cases, it is not so advanced quantitatively as the stationary wave theory.

Instrumental work in America has been concerned with the development of the strain seismographs already mentioned (Benioff 1955). Sharply tuned seismographs have also been developed to facilitate the isolation of microseisms from different sources; this is described by Donn, Ewing & Press (1954).

A description of work carried out in other countries is outside the scope of this article but mention might be made of the work of M. Båth in Sweden. He has been studying for several years the microseisms recorded at Upsala which he has also compared with those at Scoresby Sund and Bergen. He concludes that the Upsala microseisms are mainly local in origin, being due to the passage of meteorological fronts across the Norwegian coast. He thinks the mechanism of formation could be the stationary wave process in some instances and the Press-Ewing process in others. The refraction diagrams given by Iyer & others (1958) show that no appreciable microseism energy could penetrate into Scandinavia from Atlantic depressions and this must explain to some extent the very localized effects found by Båth.

Mention has already been made of Bernard who has been using mobile seismic stations and taking many records at different places in France. He was the first to observe the two to one ratio between waves and microseisms, and his later work has confirmed this. Another Frenchman, Imbert, has made a study of microseisms recorded in Antarctica.

In conclusion it can be stated that the study of microseisms in Britain is undergoing a revival in the light of recent developments where full use of modern instrumentation has been made to study the physics of the problem in detail. It is hoped that the fresh interest will be shared by other nations and full use made of these tiny oscillations as an aid to meteorology, oceanography, and geophysics.

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Geophysical Discussion of the Royal Astronomical Society

A discussion was held in the rooms of the Royal Astronomical Society on 1958 January 17 at 16h 15m on **Chemical and Radiometric Methods of Surveying,** under the chairmanship of Sir William Pugh. He remarked on the increasing importance of nuclear energy in our economy and on the usefulness of radiometric surveying in prospecting for uranium ores, mapping geological units, defining structures, and studying mineral deposits.

The first speaker, *Mr F. H. U. Bowie* of the Atomic Energy Division of H.M. Geological Survey, in introducing his talk on radiometric surveying, pointed out the stimulus the atomic bomb has had on prospecting for uranium and thorium ores. Before the war these elements were considered only waste products in radium extraction. Prospecting is now based on geological knowledge aided by radiation counters either of the Geiger-Müller or scintillation types. Instruments of three sorts were described and discussed: portable ones for field use, car-carried recording meters, and airborne units. Transistorized scintillometers using sodium iodide crystals are now the most common and have advantages over the Geiger-Müller tubes of greater statistical accuracy and a smaller volume for the cosmic ray background. A portable instrument is chosen on the basis of how well its results are reproduceable. The Geological Survey's Geiger-Müller tube ratemeter weighs 3 lb, can detect 5 pulses per second and averages the readings over a short but variable time interval. Its reproduceability is ± 2 per cent. A probe may be used with a portable instrument for assaying in the field. An instrument mounted in a car may cover a small area but if raised to a height of $5\frac{1}{2}$ ft, the area increases to 50 ft in diameter. The fall-off of radiation is as distance squared to a height of about 100 ft but is linear above this due to gamma-ray scatter in the atmosphere. Airborne instruments are usually flown at an altitude of 500 ft and cover an area in lane-widths of $\frac{3}{4}$ mile though for thorough coverage 1,000 ft is preferable. Slides were shown giving results of surveys: one compared portable counters over the Dolgelly slates and aluvium and a set compared the isorads (line of equal radiation) obtained with different flight spacings on an airborne survey of one anomalous area. Bowie closed with comments on the geochemical methods of prospecting for uranium and thorium ores employed in the Colonies. There, when searching for uranal oxides in surface waters, ion exchange resin, colorimetric, or chromatographic, analytical methods are used.

Dr J. S. Webb of the Department of Geology, Imperial College, continued with the topic of geochemical prospecting. This aims at locating areas where an element occurs in anomalous amount relative to its average abundance in the Earth's crust. The primary dispersion pattern of such an anomaly relates to the physico-chemical processes forming it and includes mineralization and any alterations accompanying it. As these may involve the travel of solutions a considerable distance along channels, there may be difficulty in locating the source of the primary dispersion pattern. A secondary dispersion pattern may result from weathering, both chemical and mechanical. Such a dispersion may be residual or may be transported by ground and surface waters, soil creep, or biological activity.

Dilution presents difficulties for using the analysis of water to detect an anomaly, and the study of stream sediment may be more fruitful. (Slides were projected showing the Cu, Co and Ni contained in residual soil over a deposit in Uganda, and copper analyses of sediment collected along a stream bed.) Plant analyses may reflect the plant's discrimination in absorbing elements from the soil and therefore analysis of the soil gives more reliable information. Sensitivity, simplicity and speed are the requirements of geochemical methods. Chromatographic and colorimetric techniques are the most common, and spectrography and fluorimetry have been used in special cases. Geochemical methods are used a great deal for mineral prospecting in Canada and Africa.

The third speaker was *Dr J. E. T. Horne* of the Atomic Energy Section of H.M. Geological Survey. He spoke on age determination by the products of radioactive decay. This may be important in the study of relative age relationships in ore genesis. A slide showed how the decay of potassium-40 had been used in establishing the epigenetic relationship of an ore from the Colorado Plateau. The main emphasis was on uranium–lead–thorium methods. Chemical analysis for these three elements plus an isotope study of lead minerals permits the calculation of several isotope ratios. Discrepancies may result from loss of material prior to the analysis and this be controlled by factors such as grain size or chemical. stability. A broad programme of study is often needed to assess these difficulties.

Dr W. H. Bullerwell of the Geophysics Branch of H.M. Geological Survey was the last to speak and his topic was the use of radioactive logging in boreholes. A slide comparing one such record with others (self-potential, resistivity, hole diameter, and lithology) made from studies in the same borehole, served to introduce the subject and underline an argument for the correlation of several types of logs in making deductions from borehole data. Methods of radioactive logging reviewed included those for recording gamma rays and neutrons. The gamma ray detection was by the methods Mr Bowie had outlined. A scintillation spectrometer which can give information about the energy of the radiation as well as its rate of reception may be useful in determining what elements are its source. If a source of neutrons is lowered down a borehole either the secondary neutrons or gamma rays, emitted from the walls of the borehole by bombardment from this source, may be recorded. A gamma–gamma log may give a measure of the electron density which scatters back the gamma-rays from a gamma-source. Development of this in conjunction with a scintillation spectrometer is now in progress.

In the discussion which followed, *Dr D. T. Germain-Jones* asked Dr Webb about the possibility of geochemical prospecting methods being applied to oil prospecting. Dr Webb replied that this seemed particularly difficult due to the formation of hydrocarbons in the soil from organic matter and not as likely of success as in the search for minerals deposit. A question to Dr Webb about the use of helicopters in some geochemical prospecting work drew a reply that this was for the transport of personnel. *Dr V. Armstrong* asked Mr Bowie about the cost of counters. He said that portable Geiger-Müller counters cost £45 and scintillometers about £75. *Mr W. A. Willox* asked if portable counters were much used while making traverses during regional geological surveying. In reply Mr Boyle said the use of a scintillometer, weighing only 5 lb by virtue of the use of transistors, was being investigated. This might be sensitive enough to contrast geological formations but a few feet of overburden could mask the effect.

The Chairman thanked the speakers and the contributors to the discussion and adjourned the meeting at 18h 05m.

Book Review

The Planet Earth

Edited by D. R. Bates, F.R.S.

312 pp. 35s. net. (1957, Pergamon Press)

The art of lecturing to, or writing for, an educated audience who know little or nothing of one's own scientific discipline, or of any at all, is usually acquired only by hard work and steady application to the waste-paper basket. The most successful lecturer or writer begins by explaining the essential facts and principles of his subject very simply and directly, avoiding all technical jargon and if he does this thoroughly, he will find that subsequently he can carry his audience with him into the most advanced ideas. But it requires a great effort of imagination to realize how little other people know of facts and ideas that form the half-unconscious background of one's own thinking; and without this effort it will not be possible to avoid jargon, to explain each step as it is taken without invoking arguments that are familiar to the specialist but to no one else, and to write throughout in that simple, straightforward English that demands no effort from the reader but every effort from the writer.

Can there be any doubt, however, that this hard work and hard thinking are worthwhile? Apart from the satisfaction of sharing with others one's own intellectual interests and the excitement of scientific study, it may be thought that scientists have a duty to explain to their patrons, who are now in large part the taxpaying public, their aims, methods and results. When, as with the IGY, the work of the scientist attracts wide publicity, the duty is imperative.

The object of this collection of surveys of geophysical studies is stated explicitly to be that of providing "the educated public with a background of knowledge which will aid them in appreciating the significance of some of the work being done". That being so, a reviewer must principally consider whether the articles are, in fact, clear, simple and straightforward, whether each step of an argument is explained as it is taken, and whether the author avoids the use of arguments and facts which are familiar to the specialist but which only confuse one who is not.

The reviewer may readily concentrate on this aspect of this book because the names of the authors of the articles are themselves guarantees of authority and of a thoroughly up-to-date picture.

In the first chapter, Professor Chapman introduces the IGY. This is followed by a discussion of the origin, age and possible ultimate fate of the Earth by Gerard P. Kuiper. Professor K. E. Bullen describes the deep interior, Professor J. Tuzo Wilson the crust, and Dr G. E. R. Deacon the oceans. The chapter on the geomagnetic field is by E. H. Vestine, while Professor Bates himself contributes that on the composition and structure of the atmosphere. Dr E. T. Eady has two chapters on climate and on the general circulation of the atmosphere and oceans. Dr E. J. Öpik writes on ice ages and Dr B. J. Mason on meteorology. Dr J. A.

Ratcliffe writes the chapter on the ionosphere, Professor Bates that on the airglow and Professor V. C. A. Ferraro one on aurorae and magnetic storms. The chapter on meteors is by Professor A. C. B. Lovell, Dr J. G. Wilson writes about cosmic rays and Professor J. B. S. Haldane discusses the genesis of life. There is a bibliography which contains anything from four to sixteen works for each article, some of them popular accounts, others significant papers on the subject. Finally, there are six appendices, to explain certain concepts more fully than is appropriate in the text.

Workers in these subjects will find each of the articles well worth reading and some of them very valuable containing as they do the latest facts and ideas hot from the mint. It is also clear from the list of chapters that almost every aspect of the physical constitution and development of the Earth is covered, the emphasis being on those subjects most directly involved in the IGY.

But as essays in popular exposition, these articles are not uniformly successful. Some are very good and while assuming little initial knowledge in the reader, introduce him to very advanced ideas. The articles of Professor Bullen and Dr. Ratcliffe are particularly successful, while Dr Eady's article on climatology gives a good idea of some of the problems of definition and analysis that arise in dealing with geophysical data.

Other chapters, however, are less satisfactory. What, for instance, will the intelligent person who is not familiar with celestial mechanics make of the statement that "Neptune's very nearly circular orbit places a million million years as the upper limit to the age of the planetary system" in Kuiper's article or one who is not an organic chemist of "The polysaccharides consist of hexagonal pyranose sugar residues linked by ether linkages in chains which may occasionally branch" in Haldane's? It is probably more than coincidence that these two are much the most speculative and are densely packed with ideas and arguments of wide range which, however, will be largely unfamiliar to anyone not widely read. Kuiper's chapter may be unfortunately placed at the beginning since it could probably be read with more understanding after some of the others.

The general reader will find some of the articles very good and others not so successful. Geophysicists and astronomers will find all the articles interesting and some most stimulating. This is a book that should be in all geophysical libraries and can be recommended for sixth-form libraries of schools.

It is well printed and the diagrams are clear and helpful although, for a book of this type, not over many.

A.H.C.

ТЕРМИЧЕСКАЯ ИСТОРИЯ ЗЕМЛИ С УЧЕТОМ ПЕРЕМЕННОЙ ТЕПЛОПРОВОДНОСТИ ЕЕ ОБОЛОЧКИ

Е. А. Любимова

Данные о распространенности радиоактивных элементов и теплопроводности Земли говорят в пользу представления, что ее начальная температура $T_0(r)$ была ниже температуры плавления. В статье представлены результаты расчетов $T_0(r)$ для Земли, образовавшейся путем объединения частиц газопылевого протопланетного облака. Рассматривается перераспределение радиоактивных элементов от однородного к расслоенному.

Вещество оболочки рассматривается как диэлектрик, обладающий в области высоких температур не только молекулярной теплопроводностью, но также и лучистым теплообменом. Для расчетов решения уравнения теплопроводности используются метод функции Грина и метод гидравлических аналогий.

Показано, что возрастание на большой грубине эффективной теплопроводности, обусловленное переносом лучистой энергии не приводит к значительной потере внутреннего тепла. Теплопроводность верхних слоев Земли обусловлена, главным образом, ее молекулярной составляющей, которая убывает с ростом температуры и поэтому препятствует значительной потере внутреннего тепла. Это ведет к заключению, что тепловую историю Земли следует рассматривать как вековое разогревание ее недр и одновременно слабое охлаждение ее внешних слоев.

СЕЙСМИЧЕСКИЕ НАБЛЮДЕНИЯ ВЗРЫВОВ АТОМНЫХ БОМБ В АВСТРАЛИИ, В 1956 ГОДУ

Б. А. Болт, А. А. Дойл и Д. Дж. Саттон

Рассмотрены сейсмические наблюдения проистекающие от четырех взрывов британских атомных бомб в Маралинге (Южная Австралия), в 1956 г. Обе фазы, как *P* так и *S*, регистрировались на полевых станциях вдоль трансавстралийской железной дороги, для которых $0{,}4° < \Delta < 11°$, а также на сейсмографе системы Бениова вблизи Аделаиды, для которого $\Delta = 8°$.

Сроки передвижений указывают, что скорость $P_n = 8{,}21 \pm 0{,}005$ км/сек, а скорость $Sn = 4{,}75 \pm 0{,}01$ км/сек. Имеются также указания на то, что скорости фаз *P* и *S*, пути продвижения которых находятся вблизи поверхности, соответственно достигают $6{,}03 \pm 0{,}09$ км/сек и $3{,}55 \pm 0{,}04$ км/сек. Никаких вынужденных колебаний, возникающих благодаря прохождению через промежуточные слои, не наблюдалось.

Наблюдения можно объяснить предположив, что существует всего один слой коры, в котором сейсмические скорости постоянны. По этой гипотезе толщина коры получается в 32 км по фазе *P* и в 39 км по фазе *S*.

В докладе разбираются более подробно результаты взрывов в Маралинге, рассмотренные прежде Дойлом. На основании этих данных был опубликован первый труд по изучению структуры коры со стороны Австралии, с применением метода преломления.

ПЕРЕМЕЩЕНИЕ ПОВЕРХОСТИ ОДНОРОДНОГО УПРУГОГО ПОЛУПРОСТРАНСТВА, ПРОИСХОДЯЩЕГО БЛАГОДАРЯ ПОДЗЕМНОМУ ВРАЩАЮЩЕМУ ИМПУЛЬСУ

Х. Л. Пекерис и И. М. Лонгман

Исследуется передвижение поверхности однородного, упругого полупространства, происходящего благодаря вращающему импульсу, точка приложения которого внутри структуры. Подразумевается, что ось вращающего момента параллельна поверхности и изменение в зависимости от времени последнего соответствует функции H(t) формулы Хевисайда. Результаты сопоставляются с решением Пиннея (1954 г.), решавшего ту

же задачу иным методом. Основное свойство, представляющее интерес для физиков, обнаружено в областях где непосредственной волне *S* предшествует преломленная волна *SP*. Найдено, что в этих случаях смещение на поверхности начинается с *бесконечной* амплитуды во время появления волны *SP*, а что за ней следует еще более выраженная *бесконечность* когда поверхности достигает волна *S*. Также, для небольших областей где волна *SP* не появляется, смещение начинается с резкого импульса в виде функции δ формулы Дирака. Кривые Пиннея не указывают ни на одно из этих свойств. Результаты исследования показаны на рис. 2, 3 и 4.

ФАКТОР ВРЕМЕНИ P ДО 30°

(Вторая Статья)

Гарольд Джеффрис

Изучается новая серия землетрясений для проверки фактора „P" для европейских землетрясений до 30°, а в особенности их состояния около 20°. Поправки таблицы автора, данные в 1939 году, хорошо совпадают с поправками на основании более раниих данных 1953 года.

Наблюдения „Pd", произведенные Лейман, соотевествуют её интерпретации, когда „Pr" нагоняет „Pd" при приблизительно 14°, но во всяком случае удивительно постояностно dt/dΔ для „Pd". Повидимому возможно, что происходит утроение последнего и что наблюдения относятся к области более отдаленной точки перегиба кривой времени.

Для области Японии не получено полезных новых данных.

Несколько землетрясений в северной и центральной Америке подтверждают данные таблицы 1939 года, в районе от 8° до 30°, с точностью до доли секунды, за исключением возможности добавочной постоянной. Дополнительные сведения о факторе времени при расстоянии менее 8°, даны в статье Лейман. Эти данные указывают, что скорости для Европы почти совпадают со скоростями для северной Америки, но что факторы времени для американских землетрясений от 8° до 30°, относительно 5°, оказываются около двух секунд короче европейских.

ПОЛЗУЧЕСТЬ ГОРНЫХ ПОРОД И ТЕРМИЧЕСКАЯ НЕУСТОЙЧИВОСТЬ

Гарольд Джеффрис

Теорию неустойчивости нагретой снизу жидкости можно легко применить к горным породам, удовлетворяющим видоизмененному закону ползучести Ломница.

Найдено, что неустойчивость невозможна, а следовательно не возникают конвекционные потоки.

О РАВНОЦЕННОСТИ ПРИБЛИЖЕНИЙ, ПОЛУЧЕННЫХ МЕТОДОМ ПЕРЕВАЛА И ЛУЧЕВОЙ ТЕОРИИ В ПРОБЛЕМАХ УПРУГИХ ВОЛН

С. Д. Чопра

Известно, что метод Бромича, согласно которому формальные решения проблем упругих волн в слоистых средах, полученные методом теории волн, разлагаются в серию отрицательных экспоненциальных членов, дает отраженные и преломленные волны, предсказанные лучевой теорией геометрической оптики.

Показано на примере, что оценка членов такого разложения методом перевала дает точные выражения для смещений отраженной и преломленной волн, полученных из геометрической теории. Использованный пример—это упругие волны из гармонического точечного источника расходящихся волн, находящегося посредине внутреннего слоя, расположенного между полупространствами с одними и теми же упругими свойствами: идеальной упругостью, однородностью и изотропией.

В разборе даются также коэффициенты амплитуд смещений в отраженной и преломленной волнах, в том случае, когда волна плоскости P или SV падает на пересечение двух упругих сред. Эти коэфициенты в том виде, в каком они встречаются в решениях задач отражения и преломления волн на пересечениях плоскостей.

Geophysical Journal

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